such that $P(f_1, \dots, f_t) = 0$ is $P \equiv 0$.

Ex: $\mathrm{trdeg}_{\mathbb{C}} \mathbb{C} = 0$

$\mathrm{trdeg}_{\mathbb{C}} \mathbb{C}(x) = 1$ ( $1/x, x$ alg. dep )

$\mathrm{trdeg}_{\mathbb{C}} \mathbb{C}(x_1, \dots, x_n) = 1$

- $X$ irreducible, $\dim X = \mathrm{trdeg}_{\mathbb{C}} K(X)$

- $X$ reducible, $X = Y_1 \cup \dots \cup Y_m$

  $Y_i$ irreducible components

  $\dim X = \max\{\dim Y_i\}$

—

$\mathbb{A}^n \hookrightarrow \mathbb{P}^n$, $T \subset \mathbb{C}[x_0, \dots, x_{n+1}]$

such that $\forall F \in T$, $F$ is homogeneus

$V(T) = \{p \in \mathbb{P}^n / F(p) = 0 \ \forall F \in T\}$

$\rightarrow$ Zariski topology defined analogously

$H_i = V(x_i)$ hyperplane is closed, $i = 0, 1, \dots, n$

$U_i = \mathbb{P}^n \setminus H_i$ OPEN

$H_i = V(x_i) \cong \mathbb{P}^{n-1}$

$U_i = \mathbb{P}^n \setminus H_i \cong \mathbb{A}^n, \quad \mathbb{P}^n = \bigcup_{i=0}^{n} U_i$

Fix some $i$

$$\begin{tikzcd} \mathbb{A}^n \arrow[r, hook', "\cong"] & U_i \subset \mathbb{P}^n \\ \text{affine } X \arrow[u, hook] \arrow[r, hook] & X \subset \overline{X} \arrow[u, hook] \end{tikzcd} \quad \left[ \begin{array}{l} \overline{X} \text{ projective} \\ \text{closure of } X \end{array} \right]$$

<u>Remark</u>. $\mathbb{P}^n$ is the closure of $U_i$

(i.e. the smallest closed subset containing $U_i$)

<u>Fact</u>: $\dim X = \dim \overline{X}$

$\dim \mathbb{P}^n = \dim \mathbb{A}^n = n$

<u>PROPOSITION 1</u>. $X \subset Y \subset \mathbb{A}^n$

$X, Y$ irreducible affine. Then $\dim X \leq \dim Y$ and equality holds iff $X = Y$.

$\dim X := \operatorname{trdeg}_{\mathbb{C}} K(X) =: \operatorname{trdeg}_{\mathbb{C}} K[X]$

$K(X) =$ quotient field of $K[X]$

Proof: $Y = \mathbb{A}^n$

$$K[X] = \frac{\mathbb{C}[x_1, \ldots, x_n]}{\mathcal{I}(X)} = \frac{K[Y]}{\mathcal{I}(X)}$$

$\rightarrow \dim X \leq \dim Y$

Have to show that if $X \subsetneq Y$ then $\dim X < \dim Y$. By contradiction suppose $\dim X = n$

$\Longrightarrow \exists f_1, \ldots, f_n \in \mathbb{C}[x_1, \ldots, x_n]$ algebraically independent such that $\underbrace{\bar{f}_1, \ldots, \bar{f}_n}_{\subseteq K[X]}$ are alg. indep.

$\exists g \in \mathcal{I}(X) \setminus \{0\}$, note that $g, f_1, \ldots, f_n$ are alg. dependent.

$\Longrightarrow \exists P \in \mathbb{C}[t_1, \ldots, t_{n+1}]$ such that $P(g, f_1, \ldots, f_n) = 0$ (we may take $P$ irred.)

$$\Downarrow$$

$P(\bar{g}, \bar{f}_1, \ldots, \bar{f}_n) = 0$, $\bar{g} = 0$ $(g \in \mathcal{I}(X))$

$\quad\;\, \|$

$P(0, \bar{f}_1, \ldots, \bar{f}_n) \Rightarrow \bar{f}_1, \ldots, \bar{f}_n$ are alg. dep

\#

<u>PROPOSITION 2</u>: Let $F \in \mathbb{C}[x_0, \dots, x_n] \setminus \mathbb{C}$ (homogeneus). $\dim \mathcal{V}(F) = n-1$

<u>Proof</u>: By prop. 1 $\dim \mathcal{V}(F) \leq n-1$

Denote $V := \mathcal{V}(F)$.

Induction on $n$ <u>$n=1$</u>

$V$ is a point in $\mathbb{P}^1 \to \dim V = 0$

($p$ a point $\to K[p] = \mathbb{C}$)

<u>$n \geq 2$</u>: If $V = H_n \cong \mathbb{P}^{n-1}$ O.K.

So we may assume $V \neq H_n$

$W := V \cap H_n$

$G(x_0, \dots, x_{n-1}) = F(x_0, \dots, x_{n-1}, 0) \neq 0$

and is not constant (homog.) $V \neq H_n$

$\Rightarrow G$ has positive degree

Hence $W := V \cap H_n = \mathcal{V}(G) \neq \emptyset$ and

by induction $\dim W = n-2$

We have $W \subsetneq V \subsetneq \mathbb{P}^{n-1}$ and by prop. 1

$\dim W < \dim V < n \Rightarrow \dim V = n-1$ $\boxtimes$

$\{\text{Curves of degree 4 in } \mathbb{P}^2\} = \mathbb{P}^{14}$

$\uparrow$

$S = \{\text{singular curves}\}$

<u>Claim</u> $\dim S = 13$

Fix $p \in \mathbb{P}^2$

$S_p = \{V \in S : V \text{ is singular at } p\}$

$F$ hom. of degree 4, $V = \mathcal{V}(F)$

$V$ is singular at $p \longleftrightarrow$

$$\underbrace{\frac{\partial F}{\partial x_0}(p)} = \underbrace{\frac{\partial F}{\partial x_1}(p)} = \underbrace{\frac{\partial F}{\partial x_2}(p)} = 0$$

hypersurfaces

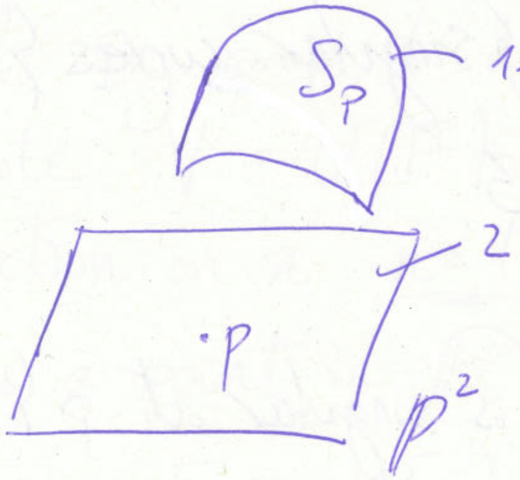
(each one takes the dimension down by 1)

$$\mathbb{P}^{14} \supset S_p = \mathcal{V}\Big(\underbrace{\frac{\partial}{\partial x_0}(p)}, \underbrace{\frac{\partial}{\partial x_1}(p)}, \underbrace{\frac{\partial}{\partial x_2}(p)}\Big)$$

polynomials in the coefficients

$\Longrightarrow \dim S_p = 14 - 3 = 11$

$$\Rightarrow \dim S = 13 = \dim S_p + \dim \mathbb{P}^2 =$$
$$= 11 + 2$$




$$S = S_1 \underset{\neq}{\supset} S_2 \underset{\neq}{\supset} \underset{\substack{\parallel \\ \{\text{set of curves with at least 3 sing. points}\}}}{S_3}$$

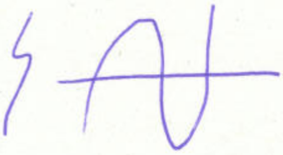
$\exists$ an open nonempty subset, $\mathcal{U}$, in $S_3$ parametrizing curves of type:

(Write the eq. of one)

$\dim S_1 = 13$, $\dim S_2 = 12$, $\dim S_3 = 11$

Is $S_3$ irreducible? NO

$\overline{\mathcal{U}} \subset S_3$ is an irreducible component of $S_3 \Longrightarrow \dim \overline{U} = 11$.

$S_3 \supset Y = \{$  $\} =$

$= \{ C \,/\, C = L \cup D,\ L \text{ a line},\ D \text{ a cubic} \} =$

$= \{ V = V(F) : F = F_1 \cdot F_3 \quad \deg F_i = i \}$

$\dim Y = 9 + 2 = 11$

$\Rightarrow S_3 = \overline{U} \cup Y$