Лекция 2-3

$A^n = \mathbb{C}^n \qquad R = \mathbb{C}[x_1, \dots x_n]$.

$\mathcal{C} = \{\mathcal{Z}(T) \quad \forall T \subset R\}$. - affine closed

$\mathcal{U} = \{A^n \setminus \mathcal{Z}(T) \quad \forall T\}$ - open subsets

$p \in A^n$, $p = (e_1, \dots e_n)$, $p = \mathcal{Z}(x_1 - e_1, \dots x_n - e_n)$.

Theorem (Weak Hilbert Nullstellensatz). - There is a bijection between points in A^m and maximal ideals in R, given as follows: $p = (e_1, \dots e_n) \longleftrightarrow (x - e_1, \dots x - e_n) \subset R$.

Proof:

Fact: $I \subsetneq R$, I is maximal $\Leftrightarrow$ R/I is a field

Part 1: The ideal $(x-a) \subset \mathbb{C}[x]$ is maximal.

$\mathbb{C}[x] \xrightarrow{v} \mathbb{C}$. $\qquad v$ is onto

$f(x) \longmapsto f(a)$

$\underset{\text{goal.}}{\ker(v) = (x-a)} \Rightarrow \mathbb{C} \cong \mathbb{C}[x]/(x-a) \Rightarrow (x-a)$ is maximal.

Part 2: goal: every maximal ideal $M \subset R$ is $M = (x-a)$ $(\exists a \in \mathbb{C})$.

The ring $\mathbb{C}[x]$ is P.I.D ($\forall$ ideal I can be generated by 1 el.)

$\exists g \in \mathbb{C}[x]: M = (g)$.

$\mathbb{C}$ is alg. closed $\Rightarrow \exists a \in \mathbb{C}: g(a) = 0. \Rightarrow g(x) = (x-a)q(x)$, q-polynomial, $\deg q < \deg g$.

The ideal $(x-a) \supset (g)$. $\Rightarrow$ they are equal!

$(x-a)$: a maximal idial; (g): a maximal by hyp.

General case, Part 1: $\forall m$ goal: $(x_1 - a_1, \dots x_m - a_m)$ is maximal in R.

$\mathbb{C}[x_1, \dots, x_m] \xrightarrow{v} \mathbb{C}$. $\qquad v$ is onto

$f(x_1, \dots x_m) \longmapsto f(a_1, \dots a_m)$.

Claim: $\ker v = (x_1 - a_1, \dots x_m - a_m)$, $\quad a_i = 0 \;\; \forall i = 1 \dots m$.

$g \in \ker v$, $\quad g = \sum_{i=1}^{m} c_i x_i + \underset{\text{monom.}}{\text{higher}}$ degree $= \sum_{i=1}^{m} x_i h_i(x_1, \dots x_m) \Rightarrow$

$\Rightarrow g \in (x_1, \dots x_m)$.

Change coord.: $x_i' = x_i - a_i$.

Part 2: $\forall$ max. ideal $M \subset R$ is of type $M = (x_1 - a_1, \dots x_m - a_m)$.

$R/M = K$ - a field.

$$\begin{tikzcd} R \arrow[r, "\pi"] & R/M = K \\ \mathbb{C}[x_1] \arrow[u, hook] \arrow[ur, "\pi_1"'] & \end{tikzcd}$$

fix x_1.

<u>Claim</u>: π_1 is <u>not</u> injective.

b.c. π_1 is injective $\Rightarrow$

Diagram of $\mathbb{C}$-vect spaces.

inf. dim. $\rightarrow$ $\mathbb{C}[x_1] \hookrightarrow K = \mathbb{C}[x_1, \dots x_m]/_M$.

Quotient field of $\mathbb{C}[x_1]$. $\longrightarrow$ $\mathbb{C}(x_1) = \left\{ \frac{f(x_1)}{g(x_1)}, \dots \right\}$

$$\begin{tikzcd} & \mathbb{C}[x_1, \dots x_m] \arrow[dr, "\pi"] & \\ \mathbb{C}[x_1] \arrow[ur] \arrow[rr, hook, dashed, "\pi_1"] \arrow[dr] & & \mathbb{C}[x_1, \dots x_m]/_M = K \\ & \mathbb{C}(x_1) \arrow[ur, dashed] & \end{tikzcd}$$

$\dim_{\mathbb{C}} \mathbb{C}[x_1, \dots x_m] = \# \mathbb{N}$ (is countable) $\Rightarrow \dim_{\mathbb{C}} K$ is countable

<u>Claim</u>: $\dim_{\mathbb{C}} \mathbb{C}(X) = \# \mathbb{C}$.

$\mathcal{G} = \{ \frac{1}{x-a} \ \forall a \in \mathbb{C} \}$. $\# \mathcal{G} = \# \mathbb{C}$:

Suppose $\sum_{j=1}^{t} \frac{c_j}{x - a_j} = 0$, $c_j \in \mathbb{C} \setminus \{0\}$.

$$\left| - \frac{c_1}{x - a_1} \right| = \left| \sum_{j=2}^{t} \frac{c_j}{x - a_j} \right| \leq \sum_{j=2}^{t} \left| \frac{c_j}{x - a_j} \right|.$$

Near $x = a_1$ this is not bounded ... is bounded. Contradiction! $\Rightarrow$

$\Rightarrow \dim_{\mathbb{C}} \mathbb{C}(x) \geq \# \mathbb{C} \Rightarrow$ diagram is not possible!

goal: $M \subset R$ is maximal $\Rightarrow M = (x_1 - a_1, \dots, x_m - a_m)$.

$$\begin{tikzcd} & R \arrow[r, "\pi"] & R/M = K \\ \mathbb{C}[x_1] \arrow[ur] \arrow[urr, "\pi_1"'] & & \end{tikzcd}$$

$\ker \pi_1 = (f)$ $\exists f \neq 0$, $f \neq const$.

We can choose f of minimal degree and monic.

$\exists a \in \mathbb{C} : f(a) = 0 \Rightarrow f = (x-a) g(x)$, $\deg g < \deg f$.

$\pi_1(f) = 0 \Rightarrow \pi_1(x-a) \cdot \pi_1(g) = 0$.

K - field $\Rightarrow \left[\begin{array}{l} \pi_1(g) = 0 \\ \pi_1(x-a) = 0 \end{array} \right.$ by our choise of $f \Rightarrow$

$\Rightarrow f = x - a$.

$\Rightarrow \ker \delta_1 = (x_1 - a_1)$.
So, $x_1 - a_1 \in \ker \pi$. (diagram is com.)

$\forall i \in \{1, .. m\} \ \exists a_i : x_i - a_i \in \ker \pi. \Rightarrow (x_1 - a_1, .. x_m - a_m) \subset \ker \pi \underset{\text{by def. of } \pi}{=}$
$= M$.
So, $(x_1 - a_1, ..., x_m - a_m) \subset M$, but $(x_1 - a_1, ..)$ is maximal $\Rightarrow$
$\Rightarrow (x_1 - a_1, ..) = M$.

Def: $X \subset A^n$; $\mathcal{I}(X) := \{ f \subset R \mid f(p) = 0 \ \forall p \in X\}$. - ideal of R (triv.)

Example: $X = \{p\} \Rightarrow \mathcal{I}(p)$ is max. ideal.

Corollary of WHN: Let $f_1, .. f_n \in R$. $\mathcal{Z}(f_1, ... f_n) = \emptyset \Leftrightarrow \exists h_1, ... h_n \in R$:
$\sum_{i=1}^{n} f_i \cdot h_i = 1$.

$p \in \mathcal{Z}(f_1, .. f_s) \Leftrightarrow f_i \in \mathcal{I}(p) \quad \forall_i = 1, .. s$.
$\mathcal{Z}(f_1, .. f_n) = \emptyset \Leftrightarrow (f_1, .., f_s) \not\subset \mathcal{I}(p) \ \forall p \in A^m \underset{\text{by WHN}}{\Longleftrightarrow} (f_1, .. f_s)$ is not contained in any max. ideal.
$\Leftarrow$ - triv.
$\underset{\text{by Zorn lemma.}}{\Rightarrow} (f_1, .. f_s) = 1 \Leftrightarrow \exists h_1 ... h_s : \sum h_i \cdot f_i = 1$. - ?.?.g.

Let $\mathbb{C}^2 = \mathbb{A}^2$, $\mathbb{C}[x, y] = R$.
$I = (x^3)$, $X = \mathcal{Z}(I)$ is a line
$\mathcal{I}(X) = (x)$.

Def: Let I be an ideal $I \subset R$, the radical of I is $\sqrt{I} = \{ f \in R :$
$\exists d \in \mathbb{N} : f^d \in I\}$. I is called radical if $I = \sqrt{I}$.
Example: 1) (x) is radical in $\mathbb{C}[x, y]$.
2) A maximal ideal is radical.
(Inded, $I \subset \sqrt{I}$.)

Lemma: $\mathcal{I}(X)$ is a radical ideal for every X.
proof: $f \in \sqrt{\mathcal{I}(X)} \Leftrightarrow \exists d : f^d \in \mathcal{I}(X) \Leftrightarrow \forall p \in X \ (f(p))^d = 0 \Leftrightarrow f(p) = 0 \ \forall p \in X. \Rightarrow$
$\Rightarrow f \in \mathcal{I}(X)$.

Question?: Let I_1, I_2 be radical ideals of R.
Suppose $\mathcal{Z}(I_1) = \mathcal{Z}(I_2)$. Does it follow that $I_1 = I_2$? Yes.

Theorem: (Hillbert Nullstellensaltz): let $I \subset R$ - ideal. If $f \in R$: such that $f(p)=0$. $\forall p \in \mathcal{Z}(I)$, than $f \in \sqrt{I}$.

(In particular, the answer to the Question is yes).

Proof: Goal: $\sqrt{I} = \mathcal{I}(\mathcal{Z}(I))$.

"$\subset$" - easy. (exercise)

goal: $\mathcal{I}(\mathcal{Z}(I)) \subset \sqrt{I}$.

$I = (f_1, \dots f_2)$. Let $g \in \mathcal{I}(\mathcal{Z}(I))$ (i.e. g vanishes wherever $f_1, \dots f_r$ vanish).

(goal: $\exists d : g^d \in I$)

Consider $\mathbb{C}[x_1, \dots x_m, y] \supset R$.

$\ell = \ell(x_1, \dots x_m, y) = g \cdot y - 1$.

Let $q \in A^{m+1}$

$g(q) = 0 \Rightarrow \ell(q) = -1 \neq 0. \Rightarrow$

$\Rightarrow \mathcal{Z}(\ell, f_1, \dots f_r) = \emptyset$. By the coroll., $1 = h_1 f_1 + \dots + h_{r+1} \ell$, where $h_i = h_i(x_1, \dots x_m, y)$.

$\ell(x_1, \dots x_m, \frac{1}{g}) = g \cdot \frac{1}{g} - 1 = 0$.

$1 = f_1(\underline{x}) h_1(x, \frac{1}{g(x)}) + \dots + f_2(x) \cdot h_2(x, \frac{1}{g(x)}) + 0$.

$1 = \dfrac{f_1 \widehat{h}_1(x) + \dots + f_r \widehat{h}_r(x)}{g^d} \Rightarrow g^d = \sum_{i=1}^{r} \widehat{h}_i f_i \Rightarrow g^d \in I$. r.t.g.