Lecture 1. Izzet Coskun

How many lines intersect 4 random lines in sace?

"Objects of algebraic geometry form algebraic families".

. Projective space $\mathbb{P}^n$.

Fix $(n+1)$ dim v.s. V.

$\mathbb{P}^n = \{1\text{-dim subspace of } V^{n+1}\}$.

Let $U_i = \{[x_0 : \dots : x_n] \in \mathbb{P}^n \mid x_i \neq 0\} = \{[\frac{x_0}{x_i}, \dots, 1, \frac{x_n}{x_i}]\}$

Let $z_j := \frac{x_j}{x_i}$ - coordinates on U_i.

$U_i = \mathbb{C}^n \ (\mathbb{A}^n)$.

Note $\bigcup_{i=0}^{n} U_i = \mathbb{P}^n$. - covering of $\mathbb{P}^n$ with $(n+1)$ affine spaces

How to change coordinates?

$p = [x_0 : x_1 \dots : x_n]$ with $x_0, x_1 \neq 0$.

$p \in U_0 \cap U_1 \;\; \overset{w_0 \;\; w_2 \dots w_n}{=} (\frac{x_0}{x_1}, 1, \frac{x_2}{x_1} \dots)$

$\overset{\shortparallel}{z_1 \dots z_n}$ $(1, \frac{x_1}{x_0}, \dots)$

$w_2 = \frac{z_2}{z_1} \cdot \frac{\cancel{x_0}}{\cancel{x_0}}$, $w_j = \frac{z_j}{z_1}$ if $j \neq 0$, $w_0 = \frac{1}{z_1}$.

Example: Riemann sphere. 

Affine stratification: $\mathbb{P}^n = U_n \sqcup \{[x_0 : x_1 : \dots : x_n] \in \mathbb{P}^n\}$ with $x_0 \neq 0$, $x_n = 0$

Affine n-space ($x_n = 0$, $x_{n-1} \neq 0 \dots$)

$\{[x_0 : x_1 : \dots : x_{n-1} : 0] \setminus 0\}/\sim$.

$\mathbb{P}^{n-1} = \{[x_0 : \dots : x_{n-1} : 0]\} \sqcup \{[x_0 : \dots : x_{n-2} : 0 : 0]\}/\sim$ ($x_{n-1} \neq 0$)

$\Rightarrow \mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \dots \sqcup \mathbb{A}^0$.

n^{th} coordinat $\neq 0$. $\quad$ $x_0 \neq 0$, other $= 0$.

Homogenous polynomials define subcpaces in $\mathbb{P}^n$, but don't define functions!

• Hypersurfaces of degree d. (f - one hom. pol. of degree d).

$\left\{ \begin{matrix} \text{Hypersurf. of degree } d \\ \text{in } \mathbb{P}^n \end{matrix} \right\}$

to describe it, we should look at h. polyn. degree d, but we are not allowed to make all parametres zero, so it is a projective space.

Example: $\mathbb{P}^2$ x, y, z $\quad 2ax^2 + 2bxy + 2cy^2 + 2dxz + 2eyz + 2fz^2 = 0$.

Not allowed $a = \dots = f = 0$.

$\Downarrow$

It's $\mathbb{P}^5$!

Hypersurface by its eqn $\sum_{i_0+\dots+i_n=d} a_{i_0\dots i_n} x_0^{i_0}\dots x_n^{i_n}$

Think of eqn by coefficients $= \sum_{|I|=d} a_I x^I$.

$(a_I) \leftarrow$ not all $=0$.

If we scale by non zero const, we get some hypersurface.

dimension $\binom{n+d}{d}$.

Monomials form a basis

Bijection: mon. of degree d in $n+1$ vector spaces. $\longleftrightarrow$ Subsets of length d in $\{1, 2, \dots, n+d\}$.

$x_0^2 x_1 x_3^3 x_4^2$ degree 8 in 5-v.s.

$\updownarrow$ $\Updownarrow$

$(\underset{1}{0}\,\underset{2}{0}\,\underset{3}{1}\,\underset{4}{3}\,\underset{5}{3}\,\underset{6}{3}\,\underset{7}{4}\,\underset{8}{4})$. subset of length 8 of $\{1, \dots 12\}$

number of times i appears is the power of x_i

$\{1, 2, 4, 7, 8, 9, 11, 12\}$.

$\begin{pmatrix} 1, 2, 4, 7, 8, 1, 11, 12 \\ 1, 2, 3, 4, 5\ 6\ 7\ 8. \end{pmatrix}$

$0\ 0\ 1\ 3\ 3\ 3\ 4\ 4$

$x_0^2 x_1^1 x_2^6 x_3^3 x_4^2$.