Lecture 2

$\mathbb{P}^n = \{ 1\text{-dim. subspaces of } V^{n+1} \} = \{ [x_0 : \dots : x_n] \setminus 0 \} \big/ \begin{smallmatrix} (x_0 : \dots : x_n) \sim (y_0 : \dots : y_n) \\ \text{if } x_i = t y_i \\ t \in \mathbb{C}^* \end{smallmatrix}$

$\mathbb{P}^n = \mathbb{A}^n \amalg \mathbb{A}^{n-1} \amalg \dots \amalg \mathbb{A}^0$

Zero locus of homogeneous polynomials

Hypersurfaces of deg d in $\mathbb{P}^n$ $\longleftrightarrow$ $f = 0$ $\longleftrightarrow$ parametrized by $\mathbb{P}^N$, where $N = \binom{n+d}{d} - 1$

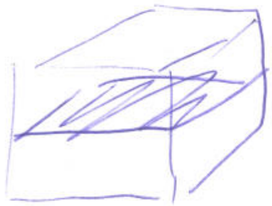
Linear space $\mathbb{P}^k \subset \mathbb{P}^n$ is the zero locus of $n-k$ independent linear homogeneous polynomials

Exs:

$\mathbb{P}^1$'s in $\mathbb{P}^2$ — lines in the plane — $ax + by + cz = 0$

$\mathbb{P}^1$'s in $\mathbb{P}^3$ — lines in space — $x = y = 0$

$\mathbb{P}^2$ in $\mathbb{P}^3$ — plane in space — $z = 0$



Grassmanian: $G(k,n) = \{ k\text{-dimensional subspaces of an } n\text{-dim. vector space } V \}$

$\Rightarrow G(1,n) = \mathbb{P}^{n-1}$

ex: • $G(2,4)$ Let $V^4 \supset W^2$ $\longrightarrow$ Complex manifold of complex dim 4

Choose bases for $W \rightarrow w_1, w_2$

$$\underbrace{\begin{bmatrix} \text{---} & w_1 & \text{---} \\ \text{---} & w_2 & \text{---} \end{bmatrix}}_{4} \Big\} 2 = \begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \end{bmatrix}$$

U_{12} – 1,2 minor is nonzero

in general U_{ij} – i,j minor is nonzero
(one of them must be nonzero)

By gaussian elimination $\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix}$

$\Downarrow$

$U_{12} \simeq \mathbb{A}^4$

When both 1,2 and 1,3 minors are nonzero, how do you change coordinates?

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix} \qquad \begin{bmatrix} 1 & \alpha & 0 & \beta \\ 0 & \gamma & 1 & \delta \end{bmatrix}$$

$$\begin{bmatrix} 1 & a \\ 0 & c \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix} = \begin{bmatrix} 1 & \alpha & 0 & \beta \\ 0 & \gamma & 1 & \delta \end{bmatrix}$$

$$= \begin{pmatrix} 1 & -\frac{a}{c} \\ 0 & \frac{1}{c} \end{pmatrix} \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\frac{a}{c} & 0 & b - \frac{ad}{c} \\ 0 & \frac{1}{c} & 1 & \frac{d}{c} \end{bmatrix}$$

$\alpha = -\frac{a}{c} \qquad \beta = b - \frac{ad}{c} \qquad \gamma = \frac{1}{c} \qquad \delta = \frac{d}{c}$

General case:

$G(k,n) \qquad W^k \subset V^n$

Choose basis W

$w_1, \dots, w_k$

$$k \Big\{ \underbrace{\begin{bmatrix} \text{---} & w_1 & \text{---} \\ & \vdots & \\ \text{---} & w_k & \text{---} \end{bmatrix}}_{n}$$

$\exists$ nonzero minor like $1, \dots, k$

$$k \Big\{ \begin{bmatrix} 1 & & 0 & \overbrace{* & \cdots & *}^{n-k} \\ & \ddots & & \vdots & & \vdots \\ 0 & & 1 & * & \cdots & * \end{bmatrix} \simeq \mathbb{A}^{k(n-k)}$$

$U_{1,2,\dots,k} \simeq \mathbb{A}^{k(n-k)}$

$\therefore$ $G(n,k)$ complex manifold of dim $k(n-k)$

Remark: Alternatively, we can think of $G(k,n)$ as

$$G(k,n) = \left\{ \begin{array}{c} \text{space of linear } \mathbb{P}^{k-1} \\ \text{in } \mathbb{P}^{n-1} \end{array} \right\} = \mathbb{G}(k-1, n-1)$$

i.e. $G(2,4)$ is also the space of lines ($\mathbb{P}^1$'s) in $\mathbb{P}^3$

$$W^k \subset V^n$$

$$\updownarrow$$

$$\underbrace{\mathbb{P}(W)}_{\mathbb{P}^{k-1}} \subset \underbrace{\mathbb{P}(V)}_{\mathbb{P}^{n-1}}$$

$G(2,4)$ $\quad V^4 = \langle e_1, e_2, e_3, e_4 \rangle$

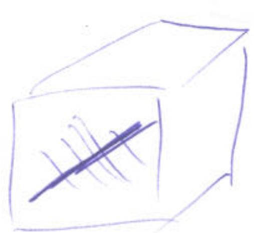
- Locus of 2-dim subspaces that don't intersect $\langle e_1, e_2 \rangle$

$W \quad w_1, w_2$

$w_1 = *e_1 + *e_2 + e_3$

$w_2 = *e_1 + *e_2 + e_4$

$$\begin{pmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix} \simeq \mathbb{A}^4 \qquad \Sigma_{0,0}$$



$\mathbb{P}^3$

- Locus W that intersects $\langle e_1, e_2 \rangle$ but does not contain e_1 or lie in $\langle e_1, e_2, e_3 \rangle$

$w_1 = *e_1 + e_2$

$w_2 = *e_1 + *e_3 + e_4$

$$\begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{bmatrix} \qquad \Sigma_{1,0}$$

- W that do not contain e_1 but are contained in $\langle e_1, e_2, e_3 \rangle$

$w_1 = *e_1 + e_2$

$w_2 = *e_1 + \quad + e_3$

$$\underbrace{\begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \end{pmatrix}}_{\mathbb{A}^2}$$

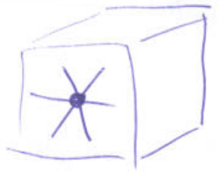
- W that contain e_1 but are not contained in $\langle e_1, e_2, e_3 \rangle$

$w_1 = e_1$

$w_2 = *e_2 + *e_3 + e_4$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{bmatrix}}_{\mathbb{A}^2}$$

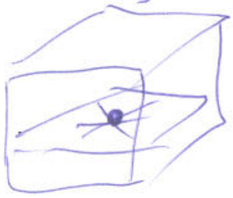
$\Sigma_{1,1}$



$\Sigma_{2,0}$

- W contains e_1
  contained in $\langle e_1, e_2, e_3 \rangle$
  but not in $\langle e_1, e_2 \rangle$

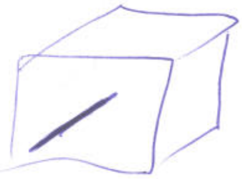
$w_1 = e_1$
$w_2 = * e_2 + e_3$ $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{pmatrix} \simeq \mathbb{A}^1$



$\Sigma_{2,1}$

- $W = \langle e_1, e_2 \rangle$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \simeq \mathbb{A}^0$



$\Sigma_{2,2}$

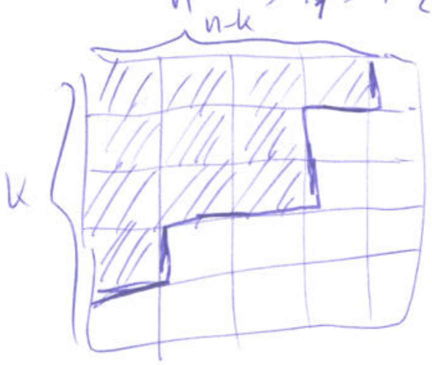
$\therefore G(2,4) = \mathbb{A}^4 \amalg \mathbb{A}^3 \amalg \mathbb{A}^2 \amalg \mathbb{A}^2 \amalg \mathbb{A}^1 \amalg \mathbb{A}^0$

General case:

$G(k,n)$

1st) Fix a partition with k parts

$n-k \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 0$



4, 3, 3, 1, 0 $\quad$ } Convention
|| $\quad$ One often omits the 0's
4, 3, 3, 1

Define $a_i = n - k + i - \lambda_i$

2nd) Fix a basis for $V = \langle e_1, \dots, e_n \rangle$

$F_1 = \langle e_1 \rangle \quad F_2 = \langle e_1, e_2 \rangle \quad \dots \quad F_i = \langle e_1, \dots, e_i \rangle$

$$\Sigma^0_{\lambda = (\lambda_1, \dots, \lambda_k)}(F_\bullet) = \left\{ W^k \;\middle|\; \begin{array}{l} \dim(W \cap F_{a_i}) = i \\ \dim(W \cap F_{a_i - 1}) = i - 1 \end{array} \right\}$$

Schubert cells

$\| \quad \Sigma^0_\lambda$

Exercise: Show $\Sigma^0_\lambda = \mathbb{A}^{k(n-k) - \sum_{i=1}^n \lambda_i} = \mathbb{A}^{\sum_{i=1}^n (a_i - i)}$

This gives a stratification of $G(k,n)$ into disjoint affine spaces

$$\Sigma_\lambda^0(F_\bullet) = \{ W^k \mid \dim(W \cap F_{\alpha_i}) \geq i \ , \ \forall 1 \leq i \leq k \}$$

← Schubert Varieties