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Lectures 3,4

Where is the description of Grassmannians as a set of zeros of a polynomial?

Exterior algebra

$V = \langle e_1, \dots, e_n \rangle$

$\Lambda^r V$ new vector space with basis

$e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_r}$ for all sets of indices $i_1 < \dots < i_r$

$\dim(\Lambda^r V) = \binom{n}{r}$ $\quad (= 0$ if $r > n)$

$\Lambda^* V = \bigoplus_{r=0}^{n} \Lambda^r V$ multiplication as in...

Example:

$(e_1 \wedge e_3) \wedge (e_2 \wedge e_4 \wedge e_5) = -e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5$

$(e_1 \wedge e_3 - e_2 \wedge e_4) \wedge (e_2 \wedge e_5 + 2 e_3 \wedge e_6)$

$= e_1 \wedge e_3 \wedge e_2 \wedge e_5 + \cancel{2 e_1 \wedge e_3 \wedge e_3 \wedge e_6} - \cancel{e_2 \wedge e_4 \wedge e_2 \wedge e_5}$
$\quad - 2 e_2 \wedge e_4 \wedge e_3 \wedge e_6$

$= -e_1 \wedge e_2 \wedge e_3 \wedge e_5 + 2 e_2 \wedge e_3 \wedge e_4 \wedge e_6$

$G(k, n) \hookrightarrow \mathbb{P}(\Lambda^k V)$ (injective)

Plücker embedding

$W \longmapsto w_1 \wedge w_2 \wedge \dots \wedge w_k$

Pick a basis for W $w_1, \dots, w_k$

We must check that the map does not depend on the choice of basis

$$G(2,4) \hookrightarrow \mathbb{P}(\Lambda^2 V) = \mathbb{P}^5$$

$$W = \langle w_1, w_2 \rangle \longmapsto w_1 \wedge w_2$$

new basis

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$w_1' \wedge w_2' = (a w_1 + b w_2) \wedge (c w_1 + d w_2)$$

$$= ad\, w_1 \wedge w_2 - bc\, w_1 \wedge w_2$$

$$= \underbrace{(ad - bc)}_{0 \neq} w_1 \wedge w_2 = w_1 \wedge w_2 \text{ in } \mathbb{P}(\Lambda^2 V)$$

$$G(k,n) \hookrightarrow \mathbb{P}(\Lambda^k V)$$

$W = \langle w_1, \dots, w_k \rangle$

$$w_i' = \sum a_{ij} w_j \qquad w_1' \wedge \dots \wedge w_r' = \det(a_{ij})\, w_1 \wedge \dots \wedge w_k$$

What is the image of the Plücker map?

$$\Lambda^2 V \qquad \sum_{i<j} a_{ij}\, e_i \wedge e_j \overset{?}{=} w_1 \wedge w_2 \quad \text{No, in general}$$

$$\dim G(k,n) = k(n-k)$$

$$\dim \mathbb{P}(\Lambda^k V) = \binom{n}{k} - 1$$

can be written just wit wedges, no sums

The image is precisely the set of completely decomposable wedges.

$$G(2,4) \hookrightarrow \mathbb{P}(\Lambda^2 V) \qquad V \xrightarrow[f]{\wedge(w_1 \wedge \dots \wedge w_k)} \Lambda^{k+1} V \qquad \ker f = W$$

$$w = p_{12}\, e_1 \wedge e_2 + p_{13}\, e_1 \wedge e_3 + p_{14}\, e_1 \wedge e_4 + p_{23}\, e_2 \wedge e_3 + p_{24}\, e_2 \wedge e_4 + p_{34}\, e_3 \wedge e_4$$

$$w \wedge w = 2\left(p_{12} p_{34} - p_{13} p_{24} + p_{14} p_{23}\right) e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

The Plücker image of $G(2,4) \hookrightarrow \mathbb{P}^5$ is the zero locus of

$$p_{12} p_{34} - p_{13} p_{24} + p_{14} p_{23} = 0$$

$L \subset \mathbb{P}^3$ To say $M \cap L = \emptyset$

$\Leftrightarrow$ $\boxed{\text{in addition } P_{34} = 0}$

Bézout's Theorem: There are 2 lines that intersect 4 general lines in $\mathbb{P}^3$

One more solution:

$\mathbb{P}^3$

L_1 $x = y = 0$

L_2 $z = w = 0$

L_3 $x - z = y - w = 0$

What is the locus of lines that intersect L_1, L_2, L_3 in $\mathbb{P}^3$?

What deg 2 polynomial contains them?

$$ax^2 + by^2 + cz^2 + dw^2 + exy + f\,xz + g\,xw + hyz + j\,yw + kzw = 0$$

$$0 = c(xw - yz)$$

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Eisenbud, Harris, 3264 and all that
Fulton, Intersection Theory
} Books for when I am a grownup

X smooth, projective variety with a stratification by affine spaces

$$X = \coprod \Sigma_i \qquad \Sigma_i \approx \mathbb{A}^{k_i}$$

Let σ_i be the symbol corresponding to Σ_i

$A^*(X) = \oplus\, \mathbb{Z}\sigma_i$ has a ring structure graded by codimension.

Satisfying

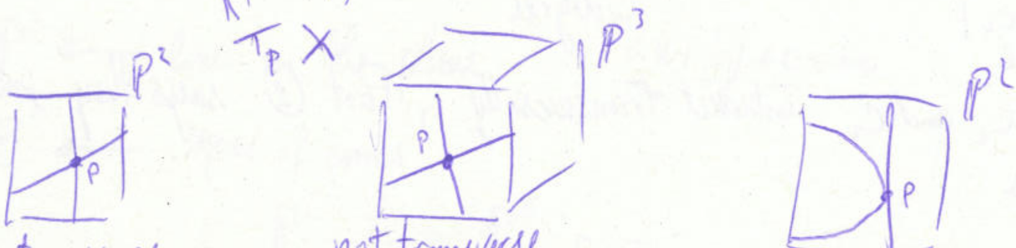
① If $Y \subseteq X$ is an algebraic set, then there is a well-defined class $[Y] \in A^*(X)$

② If $Y, Z \subseteq X$ intersecting <u>generically transversally</u>,

$$[Y]\cdot[Z] = [Y \cap Z]$$

<u>Def</u>: Y and Z intersect <u>transversally</u> at p if

$$\dim\big(\underset{\displaystyle \cap \atop \displaystyle T_pX}{T_pY \cap T_pZ}\big) = \dim T_pY + \dim T_pZ - \dim T_pX$$



transverse — not transverse (two generic lines in $\mathbb{P}^3$ don't intersect!) — not transverse

Y and Z intersect <u>generically transversally</u> if in a dense Zariski open subset of the intersection, the intersection is transverse.

$$\mathbb{P}^n = \mathbb{A}^n \amalg \mathbb{A}^{n-1} \amalg \dots \amalg \mathbb{A}^0$$

$$A^*(\mathbb{P}^n) = \frac{\mathbb{Z}[h]}{(h^{n+1})} \quad \text{where } h \text{ is the class of a linear } \mathbb{P}^{n-1}$$

$GL(n)$ acts on $G(k,n)$ by changing basis.

$\overset{\shortparallel}{n\times n}$ invertible matrices.

There is a single orbit, because we can get to any subspace V_1 from V_0 by some change of basis.

↳ the action is transitive

$\mathbb{P}^n = \{[x_0 : \dots : x_n] \setminus 0\}/\sim$

$$h = [x_n = 0] = [\sum a_i x_i = 0]$$

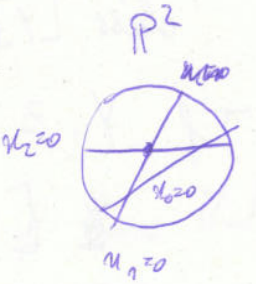
$$h^2 = [x_n = x_{n-1} = 0]$$

Example: $\mathbb{P}^2$

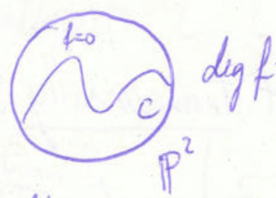
$h = [x_2 = 0]$ $\quad h = [x_1 = 0]$

$h^2 = [x_1 = x_2 = 0]$

$h^3 = [x_0 = x_1 = x_3 = 0]$



$$A^*(\mathbb{P}^2) = \frac{\mathbb{Z}[h]}{\langle h^3 \rangle}$$



$\deg f = d$

$dh \cdot eh = de h^2$ $\qquad [C] = dh$

$\underset{d}{[C_1]}\ \underset{e}{[C_2]}$ $\qquad$ ↑ degree

In case C_1 and C_2 intersect transversally, Fact ② says they intersect in de points.

$\mathbb{P}^3$ $\qquad A^*(\mathbb{P}^3) = \dfrac{\mathbb{Z}[h]}{\langle h^4 \rangle}$

$h = [x_3 = 0]$

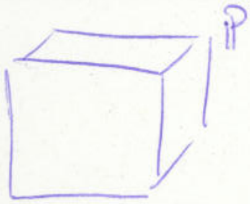
$h^2 = [x_2 = x_3 = 0]$

$h^3 = [x_1 = x_2 = x_3 = 0]$

$\mathbb{P}^3$

$[S] = dh$

$\deg f = d$



$\mathbb{P}^3$

$C = \{[x_0 : x_1 : x_2 : x_3] \mid f_1 = \ldots = f_r = 0\}$

$\deg C$ is the number of points that map to a general point

$[C] = (\deg C)\, h^2$

Space of conics in the plane $\mathbb{P}^5$

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$$

 $\mathbb{P}^2$  $\mathbb{P}^2$

(eg $xy = 0$) (eg $x^2 = 0$)

Given F_1, F_2 two random quadratic polynomials, how many in $F_1 + tF_2$ are reducible, as $t \in \mathbb{C}$ varies?

Given F_1, F_2, F_3, F_4 four random quadratic polynomials, how many in $F_1 + tF_2 + uF_3 + vF_4$ are double lines?

$\mathbb{P}^2 \leftarrow$ lines in the plane $\underline{a}x + \underline{b}y + \underline{c}z = 0$

$\mathbb{P}^5 \leftarrow$ space of conics

$$\mathbb{P}^2 \xrightarrow{\varphi} \mathbb{P}^5$$

$$[a:b:c] \longmapsto [a^2 : 2ab : b^2 : 2ac : 2bc : c^2]$$

$$(ax+by+cz)^2 = a^2x^2 + b^2y^2 + c^2z^2 + 2ab\,xy + 2ac\,xz + 2bc\,yz$$

Suppose we have a hyperplane in $\mathbb{P}^5$ that is a linear eq. in $a^2, 2ab, b^2, \ldots$ so a quadric equation in a, b, c

$$\varphi^{-1}(\text{Hyperplane}) = Y \subseteq \mathbb{P}^2$$

$$[Y] = 2h$$

$$[\varphi^{-1}(H_1) \cap \varphi^{-1}(H_2)] = 4h^2$$

2 general hyperplanes intersect the image of $\mathbb{P}^2$ under φ at 4 pts.