$$\begin{bmatrix} * & \cdots & * & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ * & \cdots & * & 0 & * & \cdots & * & 1 & & \\ \vdots & & \vdots & \vdots & & & & \vdots & \vdots & \\ * & & * & 0 & * & \cdots & * & 0 & \cdots 1 & 0 \end{bmatrix} \simeq \mathbb{A}^{\sum(a_i - i)}$$

$$\mathbb{A}^{\sum(a_i-i)} = \mathbb{A}^{k(n-k) - \sum d_i}$$

Exterior algebra

$V = \langle e_1, \dots, e_n \rangle$

$\Lambda^r V$ new vector space with basis $e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_r}$ for sets of indices $i_1 < i_2 < \dots < i_r$

$\dim(\Lambda^r V) = \binom{n}{r}$ $(= 0$ if $r > n)$

$\Lambda^* V = \bigoplus_{r=0}^{n} \Lambda^r V$ multiplication

$(e_1 \wedge e_3) \wedge (e_2 \wedge e_4 \wedge e_5) = -e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5$

$(e_1 \wedge e_3 - e_2 \wedge e_4) \wedge (e_2 \wedge e_5 + 2e_3 \wedge e_6) =$

$= -e_1 \wedge e_2 \wedge e_3 \wedge e_5 + 2e_2 \wedge e_3 \wedge e_4 \wedge e_6$

$$G(k,n) \xhookrightarrow[\text{embedding}]{\text{Plücker}} \mathbb{P}(\Lambda^k V)$$

$$W \longmapsto w_1 \wedge w_2 \wedge \cdots \wedge w_k$$

Pick a basis for W

$w_1, w_2, \ldots, w_k$

—

$$G(2,4) \hookrightarrow \mathbb{P}(\Lambda^2 V) = \mathbb{P}^5$$

$$W = \langle w_1, w_2 \rangle \longmapsto w_1 \wedge w_2$$

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

change of basis

$$w_1' \wedge w_2' = (a w_1 + b w_2) \wedge (c w_1 + d w_2) =$$

$$= ad\, w_1 \wedge w_2 - bc\, w_1 \wedge w_2 =$$

$$= \underbrace{(ad - bc)}_{\neq 0 \text{ det change of basis}} w_1 \wedge w_2$$

Therefore $[w_1 \wedge w_2] = [w_1' \wedge w_2']$

$$G(k,n) \hookrightarrow \mathbb{P}(\Lambda^k V)$$

$W \quad w_1, \dots, w_k$

$w_i' = \sum_j a_{ij} w_j$

$w_1' \wedge \dots \wedge w_k' = \det(a_{ij}) w_1 \wedge \dots \wedge w_k$

—

What is image of Plücker?

$\Lambda^2 V \quad \sum_{i<j} a_{ij} e_i \wedge e_j \overset{?}{=} w_1 \wedge w_2$ No, in general

$\dim G(k,n) = k(n-k)$ $\qquad k=2 \quad n=4 \quad 4$

$\dim \mathbb{P}(\Lambda^k V) = \binom{n}{k} - 1$ $\qquad 5$

—

Show $e_1 \wedge e_2 + e_3 \wedge e_4 \neq w_1 \wedge w_2$

—

The image is precisely the set of completely decomposable wedges.

—

$f: V \xrightarrow{\wedge (w_1 \wedge \dots \wedge w_k)} \Lambda^{k+1} V$

$\operatorname{Ker} f = W$

Therefore Plücker is injective.

$$G(k,n) \xhookrightarrow[\text{embedding}]{\text{Plücker}} \mathbb{P}(\Lambda^k V) \quad \text{injective}$$

$$G(2,4) \qquad \mathbb{P}(\Lambda^2 V) = \mathbb{P}^5$$

$$\omega = p_{12} e_1 \wedge e_2 + p_{13} e_1 \wedge e_3 + p_{14} e_1 \wedge e_4 +$$

$$+ p_{23} e_2 \wedge e_3 + p_{24} e_2 \wedge e_4 + p_{34} e_3 \wedge e_4 =$$

$$= w_1 \wedge w_2$$

What condition on p_{ij} guarantees

$$\omega \wedge \omega = 2(p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}) e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

The Plücker image of $G(2,4) \hookrightarrow \mathbb{P}^5$ is the zero locus of

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}$$

$$\left((w_1 \wedge w_2) \wedge (w_1 \wedge w_2) \underset{\Uparrow}{=} 0\right)$$

if ω can be expressed as $w_1 \wedge w_2$

$=$

Intersection $W \cap \langle e_1, e_2 \rangle \neq 0$

$\to W \quad (ae_1 + be_2) \wedge (\alpha e_1 + \beta e_2 + \gamma e_3 + \delta e_4)$

$\to p_{34} = 0$

$L \subset \mathbb{P}^3$ To say $M \cap L \neq \emptyset$

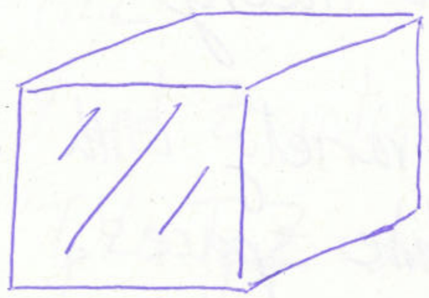
is

$\mathbb{P}^1$ $\Updownarrow$

$\langle e_1, e_2 \rangle$ in addition $p_{34} = 0$

Bezout's Thm: There are 2 lines that intersect 4 general lines in $\mathbb{P}^3$.

Another solution:



$\mathbb{P}^3$

What is the locus of lines that intersect L_1, L_2, L_3 in $\mathbb{P}^3$.

$L_1 \quad X = Y = 0$

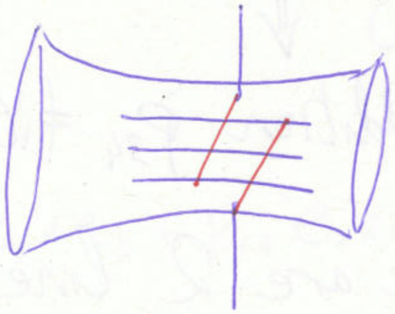
$L_2 \quad Z = W = 0$

$L_3 \quad X - Z = Y - W = 0$

What degree 2 poly's contain them

$$\cancel{ax^2} + \cancel{by^2} + \cancel{cz^2} + \cancel{dw^2} + \cancel{exy} + fxz + gxw + hyz + jyw + \cancel{kzw} = 0$$

We get: $0 = c(xw - yz)$



Bibliography

Eisenbud, Harris 3264 and all that

Fulton Intersection Theory

X smooth, projective variety with a stratification by affine spaces.

$X = \bigsqcup \Sigma_i \qquad \Sigma_i \simeq \mathbb{A}^{k_i}$

Let σ_i be the symbol corresponding to Σ_i

$\bigoplus \mathbb{Z}\sigma_i$ has a ring structure graded by codimension.

Satisfying

① If $Y \subset X$ is an algebraic set, then there is a well defined class $[Y] \in A^*(X) = \bigoplus \mathbb{Z}\sigma_i$

② If $Y, Z \subset X$ intersecting generically transversally $[Y]\cdot[Z] = [Y \cap Z]$

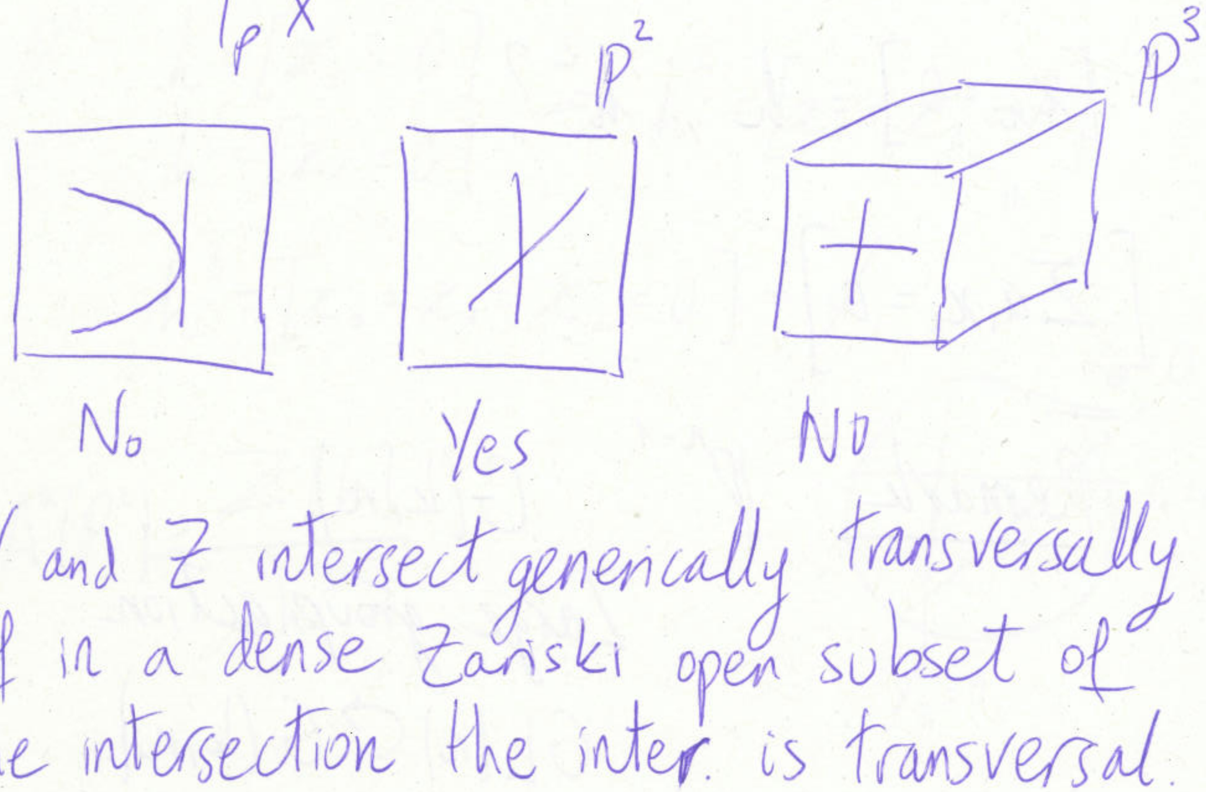
$Y \quad f_1 = \cdots = f_k = 0$

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_r} \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_r} \end{bmatrix}$$

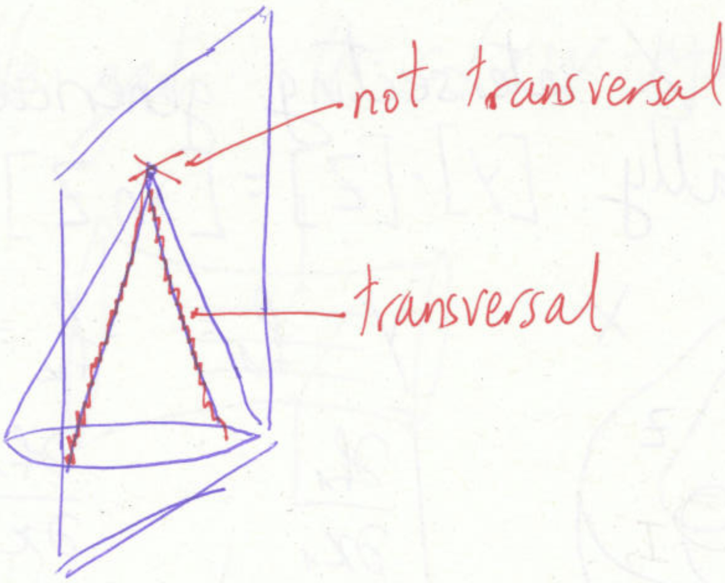
$p \in Y \cap Z$

<u>Def</u>: Y and Z intersect transversally at p

$$\dim \underbrace{T_pY \cap T_pZ}_{\subset\, T_pX} = \dim Y + \dim Z - \dim X$$



Y and Z intersect generically transversally if in a dense Zariski open subset of the intersection the inter. is transversal.



$$\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \cdots \sqcup \mathbb{A}^0$$

$$A^*(\mathbb{P}^n) = \frac{\mathbb{Z}[h]}{\langle h^{n+1} \rangle}$$ where h is the class of a linear $\mathbb{P}^{n-1}$.

$$\mathbb{P}^n = \{[x_0, \dots, x_n] \setminus \{0\}\} / \sim$$

$$[x_n = 0] = h, \quad h^2 ?$$

$$\shortparallel$$

$$\left[\sum a_i x_i = 0\right]$$

<u>Remark</u>: $\mathbb{P}^{n-1}$ $\quad G(k,n)$

Large group action

$GL(n) \circlearrowright G(k,n)$

$$\begin{array}{ccc} GL(n) & & W \\ \downarrow & & \downarrow \\ M & & k\text{-dim subspace} \end{array} \Rightarrow MW$$

This action is transitive

$$G(k,n) = {GL(n)}/{P} \leftarrow \text{stabilizer of the action}$$

$W = \langle e_1, \dots, e_k \rangle$

$$P = \begin{array}{c} k\{ \\ n-k\{ \end{array} \begin{pmatrix} \ast & \ast \\ 0 & \ast \end{pmatrix}$$

---

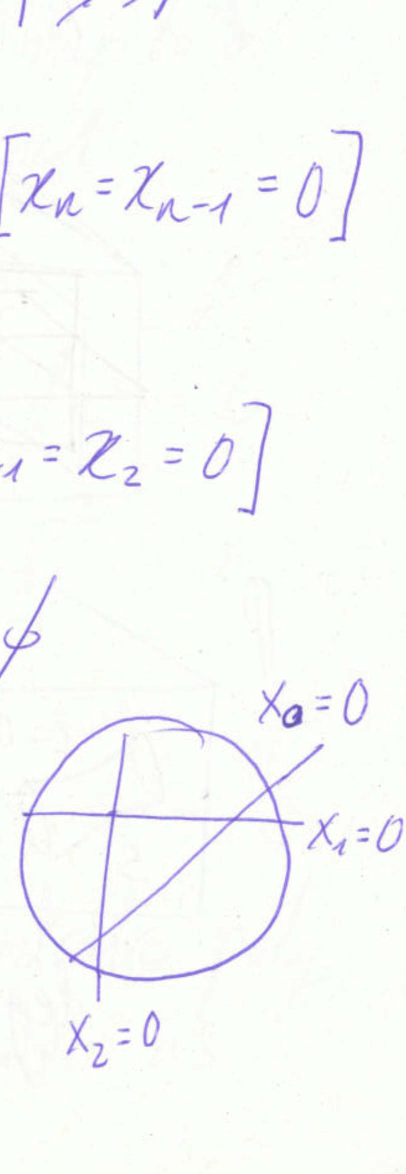
h^2

$$\begin{aligned} [x_n = 0] &= h \\ [x_{n-1} = 0] &= h \end{aligned} \Rightarrow h^2 = [x_n = x_{n-1} = 0]$$

$\mathbb{P}^2$

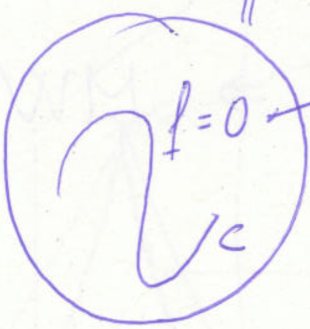
$$\begin{aligned} h &= [x_2 = 0] \\ h &= [x_1 = 0] \end{aligned} \Rightarrow h^2 = [x_1 = x_2 = 0]$$

$$h^3 = [x_0 = x_1 = x_2 = 0] = \emptyset$$



$$A^*(\mathbb{P}^2) = \frac{\mathbb{Z}[h]}{\langle h^3 \rangle}$$

$\mathbb{P}^2$



$\deg f = d$

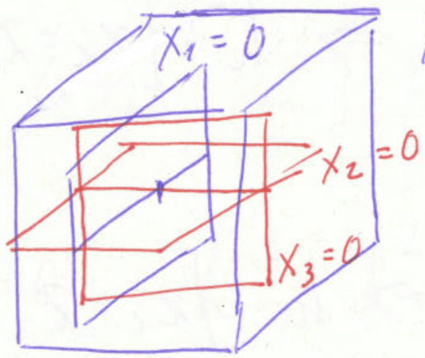
$[c] = dh$

$[c_1] \cdot [c_2] = deh^2$

$d \quad e$

$dh \quad eh$

$\mathbb{P}^3 \quad A^*(\mathbb{P}^3) = \mathbb{Z}[h] / \langle h^4 \rangle$



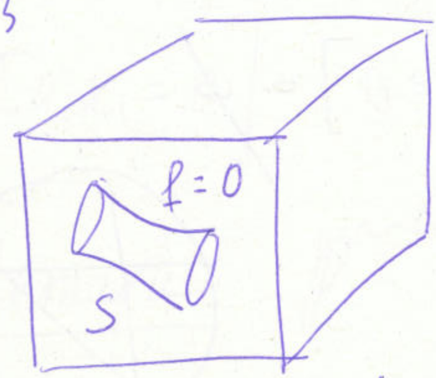
$\mathbb{P}^3$

$h = [x_3 = 0]$

$h^2 = [x_2 = x_3 = 0]$

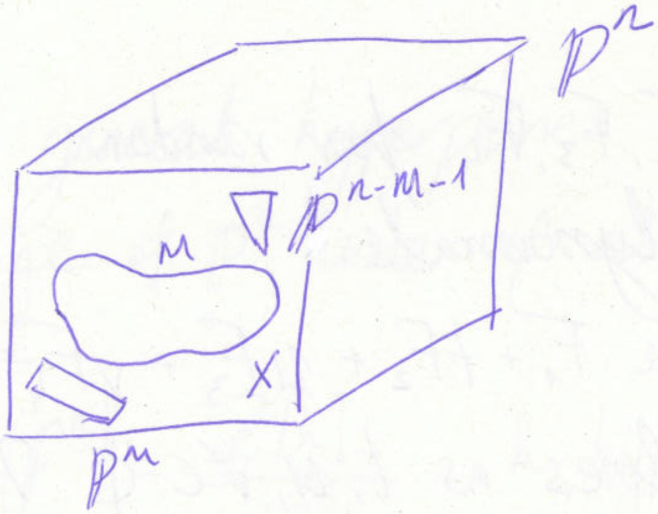
$h^3 = [x_1 = x_2 = x_3 = 0]$

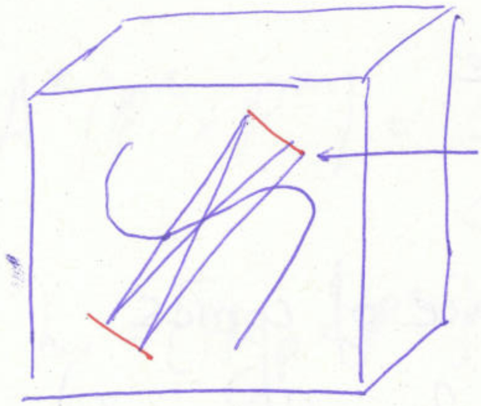
$\mathbb{P}^3$



$[S] = dh$

$\deg f = d$





$c = \{[x_0, x_1, x_2, x_3] / f_1 = \cdots = f_r = 0\}$

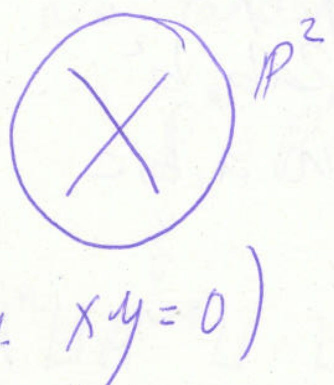
$x_2 = x_3 = 0$

deg c is the # of points that map to a general point.

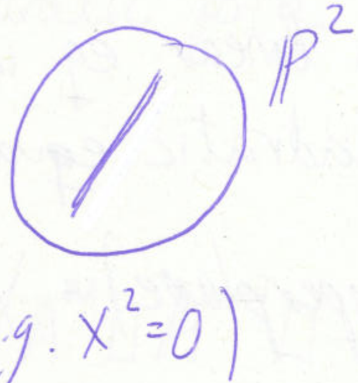
$[c] = (\deg c) h^2$

Space of conics in the plane $\mathbb{P}^5$

$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$



(e.g. $xy = 0$)



(e.g. $x^2 = 0$)

Given F_1, F_2 two random quadratic poly
How many $F_1 + tF_2$ is reducible as $t \in \mathbb{C}$ varies?

Given F_1, F_2, F_3, F_4 four random quadratic polynomials.
How many in $F_1 + tF_2 + uF_3 + vF_4$ are double lines as $t, u, v \in \mathbb{C}$? 4

$\mathbb{P}^2 \leftarrow$ lines in the plane

$$ax + by + cz = 0$$

$$\mathbb{P}^2 \xrightarrow{\varphi} \mathbb{P}^5 \leftarrow \text{space of conics}$$

$$(a,b,c) \longmapsto (a^2, 2ab, b^2, 2ac, 2bc, c^2)$$

$$(ax+by+cz)^2 = a^2x^2 + b^2y^2 + c^2z^2 + 2abxy + 2acxz + 2bcyz$$

Suppose that we have a hyperplane in $\mathbb{P}^5$ that's a linear eq in $a^2, 2ab, b^2, \ldots$
So a wadratic equation in a, b, c

$$\varphi^{-1}(\text{Hyperplane}) = Y \subset \mathbb{P}^2$$

$$[Y] = 2h$$

$$[\varphi^{-1}(H_1) \cap \varphi^{-1}(H_2)] = 4h^2$$

2 general hyperplanes intersect the image of $\mathbb{P}^2$ under φ at <u>4 points.</u>

$$A^*(\mathbb{P}^n) = \frac{\mathbb{Z}[h]}{\langle h^{n+1} \rangle}, \text{ } h \text{ hyperplane}$$

$$A^*(\mathbb{P}^n \times \mathbb{P}^m) = \frac{\mathbb{Z}[h_1, h_2]}{\langle h_1^{n+1}, h_2^{m+1} \rangle}$$

$$h_1 = [Hyp \times \mathbb{P}^n], \quad h_2 = [\mathbb{P}^n \times Hyp]$$

Q: How many conics in $Q_0 + tQ_1$ is reducible as $t \in \mathbb{C}$?

Q: How many conics in $Q_0 + tQ_1 + uQ_2 + vQ_3$ is a double line ?

Q_i are random quadratic poly

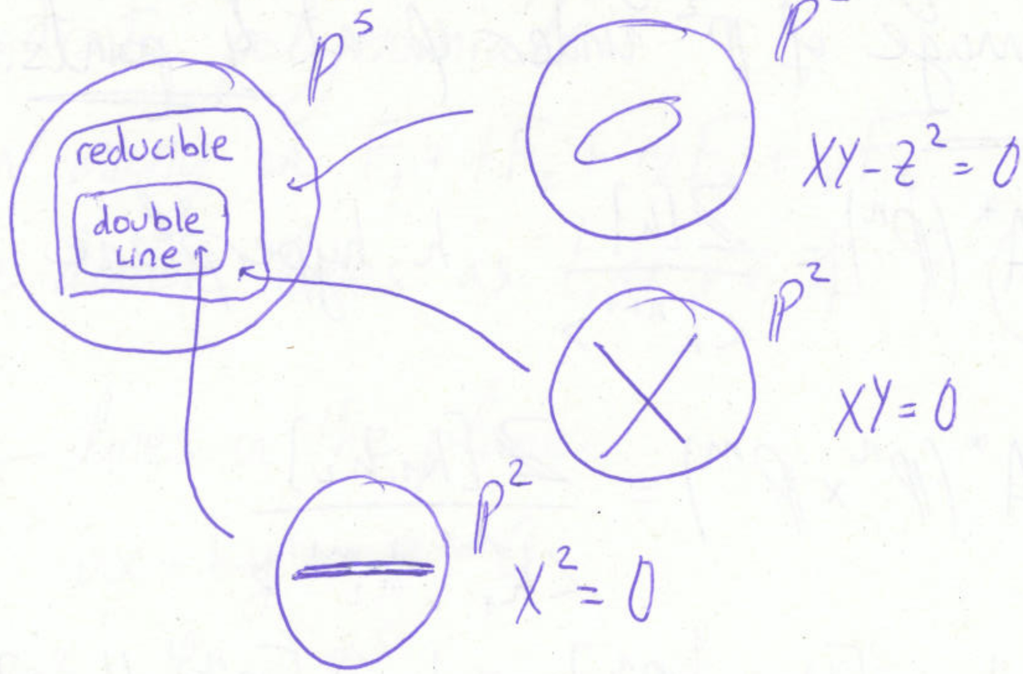
$$\mathbb{P}^n \times \mathbb{P}^m$$

$$\wr$$

$$(\mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \cdots \sqcup \mathbb{A}^0) \times (\mathbb{A}^m \sqcup \mathbb{A}^{m-1} \sqcup \cdots \sqcup \mathbb{A}^0) =$$

$$= \mathbb{A}^n \times \mathbb{A}^m \sqcup \mathbb{A}^{n-1} \cdot \mathbb{A}^m \sqcup \cdots$$

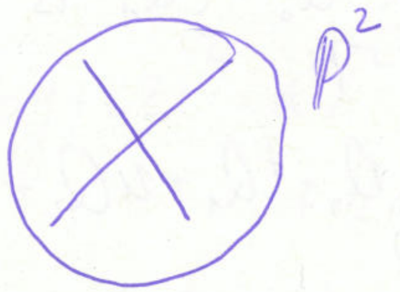
Quadratic poly in $\mathbb{P}^2 = \mathbb{P}^5$



dim {Reducible conics} = 4

[choose two lines

lines dim = 2



dim {Double conics} = 2

Questions may be written as:

Q: How many times does a line on $\mathbb{P}^5$ intersect the locus of reducible conics?

Q: How many times does a $\mathbb{P}^3$ intersect the locus of double lines?

$$\mathbb{P}^2 \longrightarrow \mathbb{P}^5$$

$$(ax+by+cz=0) \longmapsto (ax+by+cz)^2=0$$

$$(a:b:c) \longmapsto (a^2:2ab:b^2:2ac:2bc:c^2)$$

Take a hyperplane in $\mathbb{P}^5$ $\sum c_i X_i = 0$

$$\Big[\{(a:b:c)\in\mathbb{P}^2 \mid c_0a^2+2c_1ab+c_2b^2+\dots+c_5c^2=0\}\Big]$$

$$\begin{array}{c} \| \\ 2h \end{array}$$

In $A^*(\mathbb{P}^2)$ $\quad 2h\cdot 2h = \overbrace{4h^2}^{\text{class of points}}$

$\Longrightarrow$ the answer to the second question is 4.

$$\begin{array}{ccc} \mathbb{P}^2\times\mathbb{P}^2 & \longmapsto & \mathbb{P}^5 \\ & & (ax+by+cz)(dx+ey+fz) \end{array}$$

$(ax+by+cz=0)$ $\quad (dx+ey+fz=0)$

$(a:b:c)$ $\quad (d:e:f)$

$$\Rightarrow [(a:b:c),(d:e:f)] \longmapsto (ad : ae+bd : be : \dots : cf)$$

$$\sum c_i x_i = 0 \longleftarrow \text{hyperplane in } \mathbb{P}^5$$

$$c_0 ad + c_1(ae+bd) + \cdots + c_5(cf) = 0$$

$$[\{[a:b:c],[d:e:f] \,/\, c_0 ad + \cdots + c_5 cf = 0\}] =$$

$$= h_1 + h_2$$

$$(h_1+h_2)^4 \underset{\uparrow}{=} 6h_1^2 h_2^2$$

$$h_1^3,\ h_2^3 = 0$$

So the answer to the second question is $\underbrace{6/2}$ $= 3$.

$\mathbb{P}^2 \times \mathbb{P}^2$ orders the lines so each solution is counted twice.

$=$

$$ax^2 + 2bxy + cy^2 + 2dxz + 2eyz + fz^2 = 0$$

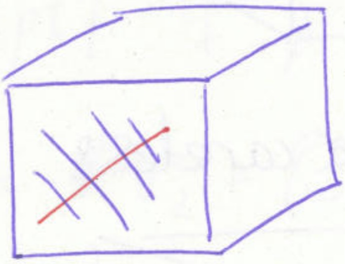
$$\updownarrow$$

$$(x\ y\ z)\begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

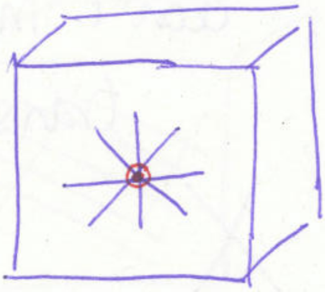
singular $\longleftrightarrow$ determinant of the matrix is zero and this det. is a degree three. $\Longrightarrow$ $\underline{3}$

Σ_1 dim 3

σ_1

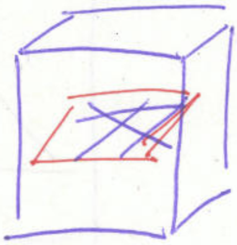


Σ_2 dim 2



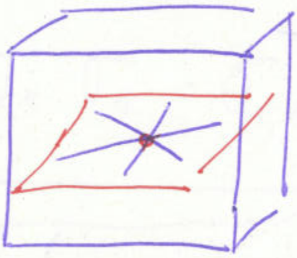
σ_2

$\Sigma_{1,1}$ dim 2



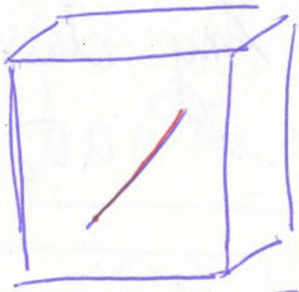
$\sigma_{1,1}$

$\Sigma_{2,1}$ dim 1



$\sigma_{2,1}$

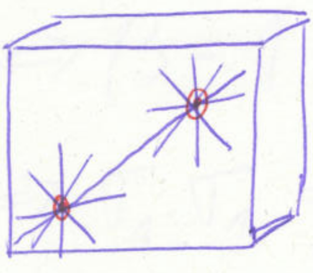
$\Sigma_{2,2}$ dim 0

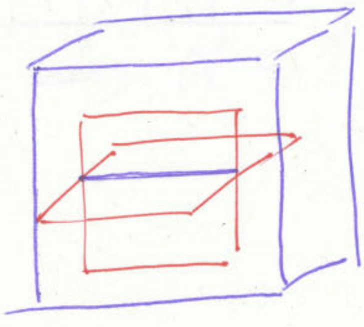


$\sigma_{2,2} = [pt]$

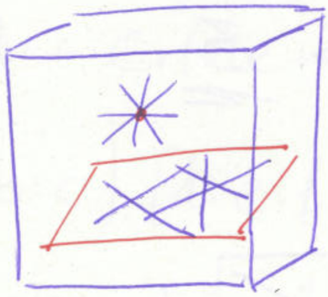
$\sigma_2 \cdot \sigma_2 = \sigma_{2,2}$

$\sigma_{1,1} \cdot \sigma_{1,1} = \sigma_{2,2}$

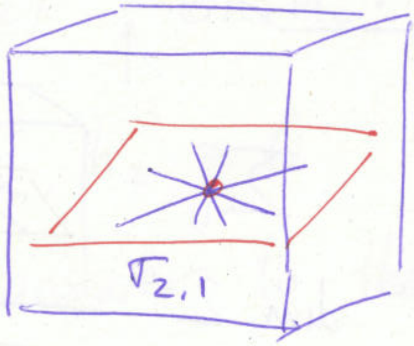




$\sigma_2 \cdot \sigma_{1,1} = 0$



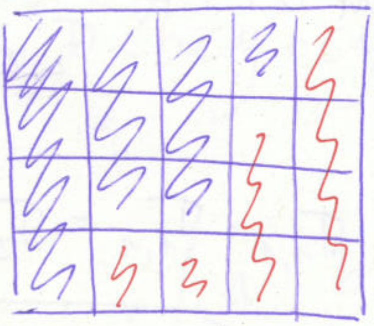
If we were careless



don't intersect transversally

Any class in codim 2

$a\sigma_2 + b\sigma_{1,1}$



$4,3,3,1 \quad \lambda$

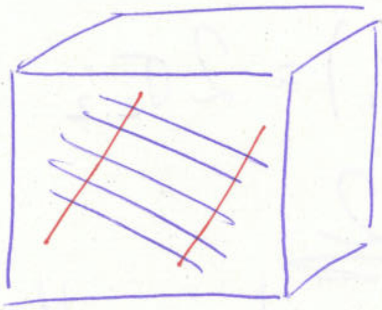
$4,2,2,1 \quad \lambda^*$ dual partition

<u>Fact</u>: Let λ, ν be partitions

$$\sum \lambda_i + \sum \nu_i = k(n-k)$$

$$\sigma_\lambda \cdot \sigma_\nu = \begin{cases} 0 & \text{otherwise} \\ [pt] & \text{if } \nu = \lambda^* \end{cases}$$

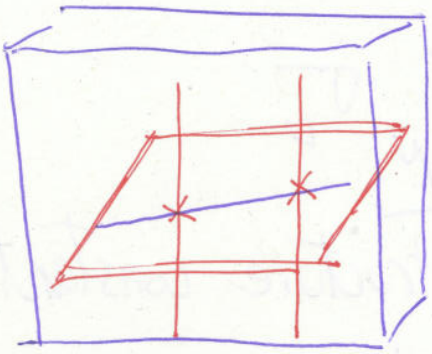
$$\sigma_1 \cdot \sigma_1 = \alpha \sigma_2 + \beta \sigma_{1,1}$$



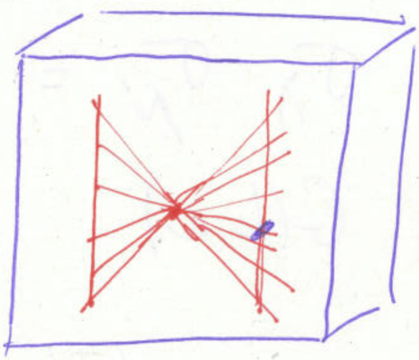
$\mathbb{P}^3$

$$\sigma_2 \cdot \sigma_1 \cdot \sigma_1 = \sigma_2 (\alpha \sigma_2 + \beta \sigma_{1,1}) = \alpha \sigma_{2,2}$$

$$\sigma_{1,1} \cdot \sigma_1 \cdot \sigma_1 = \sigma_{1,1} (\alpha \sigma_2 + \beta \sigma_{1,1}) = \beta \sigma_{2,2}$$



$\mathbb{P}^3$

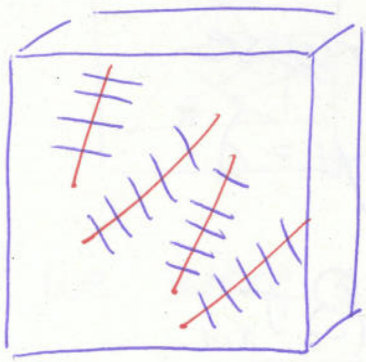


$\mathbb{P}^3$

$\Longrightarrow \beta = 1$ $\qquad\qquad \Longrightarrow \alpha = 1$

$$\Longrightarrow \sigma_1 \cdot \sigma_1 = \sigma_2 + \sigma_{1,1}$$

Q. How many lines intersect 4 lines in space? $L_1, L_2, L_3, L_4 \subset \mathbb{P}^3$



$\Sigma_1(L_1) \cap \Sigma_1(L_2) \cap \Sigma_1(L_3) \cap \Sigma_1(L_4)$

$$\sigma_1^4 = (\sigma_2 + \sigma_{1,1})(\sigma_2 + \sigma_{1,1}) = 2\sigma_{2,2}$$

$\Longrightarrow$ the answer is $\underline{\underline{2}}$

In any $G(k,n)$ one can phrase these enumerative problems and solve them in $A^*(G(k,n))$

$$\underset{G(k,n)}{\sigma_\lambda \cdot \sigma_\mu} = \sum \underbrace{c^{\nu}_{\lambda\mu}}\, \sigma_\nu$$

structure constant

Littlewood-Richardson coeff.

We have algorithms for computing them?