* = Recomended readings

2.1. An introduction to small division problems

Let X be the "phase space." , ~~and~~ $End(X)$ the set of endomorphisms of X, and $Aut(X)$, collection of automorfisms of X. If $x \in X$

$O_f(x)$ "orbit of x" $:= \{f^j(x)\}_{j \in \mathbb{N} \leftarrow \text{if endomorfism.}}^{j \in \mathbb{Z} \leftarrow \text{if automorfism.}}$

If the orbit is finite, its called periodic.

* S. Sternberg: Celestial Mechanics I (Periodicity & quasiperiodicity)

Most systems are not exactly periodic. A system is quasiperiodic, there is:

No formal definition given yet

Example: Let $X := \mathbb{R}/\mathbb{Z}$; $\alpha \in \mathbb{R}$ $\equiv R_\alpha(x) = x + \alpha$

If $\alpha \in \mathbb{Q}$ $\alpha := \frac{p}{q}$ (p, q coprimes) $\Leftrightarrow$ all orbits are periodic with period p.
$p, q \in \mathbb{Z}$

If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, R is "quasiperiodic"

$R_\alpha^j(x)_{j \in \mathbb{Z}}$ is dense in $\mathbb{R}/\mathbb{Z}$

T is an ε-approximate period of R_α if $|R_\alpha^n(x) - x| < \varepsilon_n$ $\forall x \in \mathbb{R}/\mathbb{Z}$

Completely canonically integrable systems

* "An introduction to small division problems." On-line

The problem arises in hamiltonian mechanics, in systems in which "energy" is conserved (energy in the broad sense) $\begin{cases} \dot{q}_i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H}{\partial q_i} \end{cases}$ where q_i = generalized positions
p_i = gen. momenta
H = hamiltonian

Ex: (1) $H = H(p)$ (no explicit position dependence)

(2) $q \in \mathbb{R}^n/\mathbb{Z}^n$

Because of (1): $\frac{\partial H}{\partial q} = 0 = \dot{p}_i$ $\Rightarrow$ Momenta are conserved. $\Rightarrow$

$$p_i(t) = p_i(0)$$
$$\dot{q}_i(t) = \omega_i(p) := \frac{\partial H}{\partial p_i} \Rightarrow q_i(t) = q_i(0) + \omega_i(p_i(0))\, t$$

If $n = 1$ periodic with period $T = \frac{1}{\omega(p)}$

If $n > 1$ quasi periodic in general, except if the frequency is totally resonant: $M := \{K \in \mathbb{Z}^n \mid K \cdot \omega = 0\}$

If ~~M~~ $\dim(M) = 0 \Rightarrow \omega$ non resonant

If $0 < \dim(M) \leq n - 1$

If $\dim M = n - 1 \Rightarrow$ The system is periodic.

~~Conj~~

Conjugacy & classification on dynamical systems

Let X be our space, $g, f \in End(X)$, f and g are conjugate iff $\exists\, h \in Aut(X) \,/\, f \circ h = h \circ g$