Marmi - 3

Diophantine Conditions: Given $\gamma > 0, \tau \geq 0, \alpha \in \mathbb{R}\setminus\mathbb{Q}$

$$\alpha \in D.C.(\gamma,\tau) \overset{def}{\iff} \left|\alpha - \frac{p}{q}\right| \geq \frac{\gamma}{q^{2+\tau}} \quad \forall \frac{p}{q} \in \mathbb{Q}$$

$$D.C.(\tau) = \bigcup_{\gamma>0} D.C.(\gamma,\tau)$$

$$D.C. = \bigcup_{\tau} D.C.(\tau).$$ $(\mathbb{R}\setminus\mathbb{Q})\setminus D.C.$ = Liouville numbers.

Theorem: $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ and $p(\alpha)=0$. $\deg p = m \geq 2$, $p \in \mathbb{Z}[x]$ irreducible $\Rightarrow \alpha \in D.C.(m-2)$

Proof: ~~Want to~~

$$\frac{1}{q^m} \leq \left|P\left(\frac{p}{q}\right)\right| = \left|P\left(\frac{p}{q}\right) - P(\alpha)\right| = |P'(t)| \cdot \left|\frac{p}{q} - \alpha\right| \quad \text{for some } t \in \left[\alpha, \frac{p}{q}\right]$$

If $\gamma = \sup_{[\alpha-1,\alpha+1]} \frac{1}{|P'(t)|}$, we have $\left|\frac{p}{q} - \alpha\right| \geq \frac{\gamma}{q^m}$ for large q. □

Theorem: $\ell = \sum_{n=0}^{\infty} 10^{-n!}$ is transcendental. Therefore, $\ell \in (\mathbb{R}\setminus\mathbb{Q})\setminus D.C.$

Proof: $p_n = \sum_{j=1}^{n-1} 10^{(n-1)! - j!}$, $q_n = 10^{(n-1)!}$. Then

$$\left|\ell - \frac{p_n}{q_n}\right| \leq \frac{2}{10^{n!}} = \frac{2}{q_n^n}, \text{ contradiction. } □$$

Exercise: Prove that almost all $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ is diophantine.

Hint: Get a bound on $(0,1)\setminus D.C.(\gamma,\tau)$.

Proof: $(0,1)\setminus D.C.(\gamma,\tau) = \bigcup_{\substack{p<q \\ p\perp q}} \left[\frac{p}{q} - \frac{\gamma}{q^{2+\tau}}, \frac{p}{q} + \frac{\gamma}{q^{2+\tau}}\right] \Rightarrow$

$$\mu((0,1)\setminus D.C.(\gamma,\tau)) \leq \sum_{\substack{p<q \\ p\perp q}} \frac{\gamma}{q^{2+\tau}} < \sum_{q=1}^{\infty} \frac{\gamma}{q^{1+\tau}} = \gamma\,\zeta(1+\tau)$$

Now $\mu((0,1)\setminus D.C.(\tau)) \leq \gamma\,\zeta(1+\tau)\ \forall\gamma$. Making $\gamma \to 0$, $\mu((0,1)\setminus D.C.(\tau)) = 0 \Rightarrow$

$\mu((0,1)\setminus D.C.) = 0$.

Exercise: $\forall a_m \in \{1,2\}$, show that

$$l_m = \sum_{m=1}^{\infty} a_m \cdot 10^{-m!}$$ is Liouville. In particular, Liouville are uncountable.

Theorem (Thue): $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ algebraic of degree $m \Rightarrow \alpha \in D.C.\left(\frac{m}{2}+T-1\right)$ $\forall T>0$.

(Roth): If $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, is algebraic, $\alpha \in R.T. = \bigcap_{\tau>0} D.C.(\tau)$.

Exercise: $e = \sum_{m=0}^{\infty} \frac{1}{m!} \in R.T.$

Proof: $e = [2,1,2,1,1,4,1,1,6,1,1,8,1,1,10,\dots]$

$$\left|\alpha - \frac{p_m}{q_m}\right| \gtrsim \frac{1}{q_{m+1}\cdot q_m} \quad \text{and } q_{m+1} = a_{m+1} q_m + q_{m-1}.$$

Exercise: $D.C.(\tau) = \{\alpha \in \mathbb{R}\setminus\mathbb{Q} : q_{m+1} = O(q_m^{1+\tau})\} = \{\dots \mid a_{m+1} = O(q_m^{\tau})\}$.

Linearization of an analytic map (in $\mathbb{C}$) with fixed point:

$$\mathbb{C}\{z\} = \left\{ \sum_{m=0}^{\infty} a_m \cdot z^m \;\middle|\; r = \frac{1}{\overline{\lim}\, |a_m|^{\frac{1}{m}}} > 0 \right\}$$

Suppose $f(0)=0$ with $f(z) = \lambda z + f_2 z^2 + f_3 z^3 + \dots$

we want a change of coordinates $h(z) = z + h_2 z^2 + \dots$ making f λz, that is,

$f \circ h = h \circ R_\lambda$ where $R_\lambda(z) = \lambda z$.

Def: 0 is stable if there is a neighborhood of points which stay close to 0 under iterations.

Fact: If $|\lambda| \leq 1$, then 0 is stable under iteration of $f \Leftrightarrow f$ is linearizable.

If $|\lambda|>1$ " " " " " $f^{-1} \Leftrightarrow f$ is linearizable.

Prop: If $\lambda^m \neq 1$ $\forall m \in \mathbb{Z}^+$, then f is "formally linearizable", i.e.,

$\exists!\, h \in \mathbb{C}[[z]]$ s.t. $f \circ h = h \circ R_\lambda$.

Poincaré: MNMC, tome II : "different notions of convergence".

Koenigs - 4

Proof of prop.: $RHS = h(R_\lambda(z)) = \lambda z + h_2 \lambda^2 z^2 + h_3 \lambda^3 z^3 + \dots$

$LHS = f(h(z)) = \lambda z + \lambda h_2 z^2 + \dots + \sum_{i=2}^{\infty} f_i \left(\sum_{j=1}^{\infty} h_j z^j \right)^i$ $\qquad h_1 = 1$

$$\Leftrightarrow \underbrace{(\lambda^n - \lambda)}_{\neq 0} h_n = \left[\sum_{j=2}^{\infty} f_j \left(\sum_{k=1}^{\infty} h_k z^k \right)^j \right]_{z^n}$$

$$= [z^n] \sum_{j=2}^{\infty} f_j \sum_{k_1, \dots, k_j = 1} h_{k_1} \cdots h_{k_j} \cdot z^{k_1 + \dots + k_j}$$

$$= [z^n] \sum_{|I|>1} f_{|I|} \cdot z^{\sigma(I)} \cdot \prod_{k \in I} h_k$$

$$= \sum_{j=2}^{n} f_j \sum_{\substack{k_1 + \dots + k_j = n \\ k_i \geq 1}} h_{k_1} \cdots h_{k_n}$$

$\hookrightarrow k_i + k_j \leq n \Rightarrow k_i \leq n-1 \ \forall i$

$n=2 \rightarrow f_2 h_1^2$

$n=3 \rightarrow f_2 \cdot 2 h_1 h_2 + f_3 h_1^3$

$n=4 \rightarrow f_2 (h_2^2 + 2 h_1 h_3) + f_3 \cdot 3 h_1^2 h_2 + f_4 h_1^4$

We can recursively define

$$h_n = \frac{1}{\lambda^n - \lambda} \sum_{j=2}^{n} f_j \left(\sum_{\substack{k_1 + \dots + k_j = n \\ k_i \geq 1}} h_{k_1} \cdots h_{k_j} \right). \ \square$$

How can we know whether $\sum h_n z^n$ converges (locally)?

Theorem (Koenigs - Poincaré): If $|\lambda| \neq 1$, $h \in \mathbb{C}\{z\}$.

Consider $|\lambda| = 1$, $\lambda = e^{2\pi i \alpha}$ and $\alpha \in D.C.(\tau)$, $\tau \geq 0$

Solving recurrence,

$$h_2 = \frac{f_2}{\lambda^2 - \lambda}$$

$$h_3 = \frac{1}{\lambda^3 - \lambda}\left[f_3 + 2(\lambda^2-\lambda)^{-1} f_2\right]$$

$$h_4 = \frac{1}{\lambda^4 - \lambda}\Big[f_4 + 3(\lambda^2-\lambda)^{-1} f_3 f_2 + 2(\lambda^3-\lambda)^{-1} f_2 f_3$$

$$+ 4(\lambda^3-\lambda)^{-1}(\lambda^2-\lambda)^{-1} f_2^3 + f_2^3 (\lambda^2-\lambda)^{-2}\Big]$$

we allways have the term

$$(*) = 2^{n-2} f_2^{n-1} \left((\lambda^n - \lambda) \cdots (\lambda^3-\lambda)(\lambda^2-\lambda)\right)^{-1}$$

$$|\lambda^{n+1} - \lambda| = |e^{2\pi i \alpha n} - 1| = |e^{\pi i \alpha n} - e^{-\pi i \alpha n}| = 2|\sin n\pi\alpha| \approx |\{n\pi\lambda\}|$$

$\hookrightarrow \{n\pi\alpha\} \approx 0.$

$|\{n\pi\lambda\}| \approx$

we approximate

$$|\{n\pi\alpha\}| = |n\alpha - p| \geq \frac{\gamma}{n^{\tau+1}}$$

$$|(*)| \approx 2^{n-2} f_2^{n-1} (2\gamma)^{-(n-1)} \cdot \left[(n-1)!\right]^{1+\tau}$$

This is not good enough, and that's why $|\lambda| = 1$ has problems.

Proof (Koenigs - Poincaré):

We know if $|\lambda| \neq 1$, $|\lambda^n - \lambda| \geq c > 0$. Consider $\{\sigma_i\}$ given by
$\geq ||\lambda|^n - |\lambda||$

$\sigma_1 = 1$

$\sigma_m = \sum_{2}^{m} \sum_{k_1 + \dots + k_j} \sigma_{k_1} \cdots \sigma_{k_j}$. Since f_j make $\sum f_j x^j$ convergent, we can bound f_j exponentially and so we only have to check σ_j make

$\sigma(z) = \sum_{n=1}^{\infty} a_n z^n$ convergent.

But notice that formally $\sigma(z) = z + \underbrace{\frac{\sigma(z)^2}{1-\sigma(z)}}_{\sum_2 \sigma(z)^n}$

Solving, $\sigma(z) = \frac{1+z-\sqrt{1-6z+z^2}}{4}$ (recall $\sigma(0)=0$)

we can get $\sigma_n \leq C_3 (3-2\sqrt{2})^{1-n}$.

By induction we can get h_n bounded by an exponential, and this makes h convergent.

Thm (Cremer, ~1920): If $\limsup_{n\to\infty} |\lambda^n - 1|^{-\frac{1}{n}} = +\infty$ ($\sim \|\{n\alpha\}\|^{-\frac{1}{n}}$)

$\Rightarrow \exists f(z) = \lambda z + \ldots \in \mathbb{C}\{z\}$ not linearizable.

Sketch: Build f_n s.t. f_n and $\sum f_j \sum h_{ik}$ are coincident. Then

$|h_n| \geq \frac{1}{\lambda^n - \lambda} |f_n|$