DAY 4

DIOPHANTINE NUMBERS CONDITIONS

Def: Given $\gamma \geq 0$, $\tau \geq 0$, $\alpha \in \mathbb{R}\setminus\mathbb{Q}$

$$\alpha \in DC(\gamma, \tau) \overset{def.}{\iff} \left|\alpha - \frac{p}{q}\right| \geq \frac{\gamma}{q^{2+\tau}} \quad \forall \frac{p}{q} \in \mathbb{Q}$$

- $D.C.(\tau) := \bigcup_{\gamma > 0} D.C.(\gamma, \tau)$

- $DC = \bigcup_{\tau \geq 0} DC(\tau)$

Thm (Liouville) $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ and $p(\alpha) = 0$ with $\deg p = n \geq 2$, $p \in \mathbb{Z}[x] \Rightarrow \alpha \in DC(n-2)$

Proof: We want to get $\left|\alpha - \frac{p}{q}\right| \geq \frac{\gamma}{q^n}$ $\forall \frac{p}{q} \in \mathbb{Q}$

for ~~many~~ some $\gamma > 0$.

$\left|\alpha - \frac{p}{q}\right| \leq 1$ $\qquad$ $\left|q^n p\left(\frac{p}{q}\right)\right| \geq 1$ $\qquad$ Rolle theorem

$$\frac{1}{q^n} \leq \left|p\left(\frac{p}{q}\right)\right| = \left|p\left(\frac{p}{q}\right) - p(\alpha)\right| \leq \left|\frac{p}{q} - \alpha\right| |p'(t)| \qquad \square$$

$t \in [\alpha, \frac{p}{q}]$

(so choose $\gamma = \min\left(\frac{1}{|p'(t)|}\right) = \frac{1}{\max_{t \in [\alpha, \frac{p}{q}]} |p'(t)|}$

$\rho = \sum_{n=0}^{\infty} 10^{-n!}$ is transcendental:

$q_n = 10^{(n-1)!}$

$p_n = q_n\left(10^{-1!} + \dots + 10^{-(n-1)!}\right)$

$$\left|\rho - \frac{p_n}{q_n}\right| = \sum_{m \geq n}^{\infty} 10^{-m!} \leq \frac{2}{10^{n!}} = \frac{2}{q_n^n}$$

Exercise: Prove that almost all $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ are diophantine.

Hint: 1. $(\mathbb{R}\setminus\mathbb{Q}) \cap (0,1)$

2. Get a bound on $(0,1)\setminus DC(\gamma,\tau)$

$$\mathbb{Q}\cap(0,1) = \bigcup_{q\geq 1} \bigcup_{p\in\{0,1,\dots,q-1\}} \{p/q\}$$

$$|(0,1)\setminus DC(\gamma,\tau)| \leq \sum_{q=1}^{\infty}\sum_{p=0}^{q-1} \left| \left(\frac{p}{q}-\frac{\gamma}{q^{2+\tau}}, \frac{p}{q}+\frac{\gamma}{q^{2+\tau}}\right)\right|$$

Exercise: Show that $\forall (a_n)_{n\in\mathbb{N}} \subset \{1,2\}$

$$\rho_n := \sum_{n=1}^{\infty} a_n 10^{-n!} \quad \text{is Liouville}$$

Hystory: 1909 Thue $\to$ if $\alpha\in\mathbb{R}\setminus\mathbb{Q}$ is algebraic of deg (n)

$$\Rightarrow \alpha \in DC\left(\frac{n}{2}+\tau-1\right) \quad \forall \tau > 0.$$

1955 Roth $\to$ If $\alpha\in\mathbb{R}\setminus\mathbb{Q}$, is algebraic

$$\Rightarrow \alpha \in RT := \bigcap_{\tau>0} DC(\tau)$$

Proposition: $e = \sum_{n=0}^{\infty}\frac{1}{n!} \in RT$

"Proof"

Linearization of an analytic map (in $\mathbb{C}$) with a fixed point

$$\mathbb{C}\{z\} = \left\{ \sum_0^\infty a_n z^n \;\middle|\; r = \frac{1}{\overline{\lim}\, |a_n|^{\frac{1}{n}}} > 0 \right\}$$

$$f(z) = \lambda z + f_2 z^2 + f_3 z^3 + \dots$$

$$h(z) = z + h_2 z^2 + h_3 z^3 + \dots$$

$$f \circ h = h \circ R_\lambda \qquad R_\lambda(z) = \lambda z$$

0 is stable $\overset{\text{def}}{\iff}$ it has a neighbourhood of points that stay close to 0 under iteration

<u>FACT</u>: If $|\lambda| \leq 1$ then 0 is stable under iteration of f
$\iff$ f is linearizable.

Let's try to compute h.

<u>Prop</u>: If $\lambda^n \neq 1$ $\forall n \in \mathbb{N}$, then

f is "formally linearizable"
i.e. $\exists !\, h \in \mathbb{C}[[z]]$ wich solves $f \circ h = h \circ R_\lambda$ in $\mathbb{C}[[z]]$.

<u>Proof</u>: RHS: $\lambda z + h_2 \lambda^2 z^2 + h_3 \lambda^3 z^3 + \dots = \lambda z + \sum_2^\infty h_n \lambda^n z^n$.

LHS: $\lambda(z + h_2 z^2 + \dots) + \sum_{j=2}^\infty f_j \left(\sum_1^\infty h_k z^k \right)^j$

LHS = RHS $\iff$ $(\lambda^n - \lambda) h_n = \left[\sum_{j=2}^\infty f_j \left(\sum_1^\infty h_k z^k \right)^j \right]_{\text{coef of degree } n}$

$$= \left[\sum_{j=2}^{\infty} f_j \sum_{k_1,\dots,k_j=1}^{\infty} h_{k_1} \cdots h_{k_j} z^{k_1+\dots+k_j} \right]_{\text{term of degree } n}$$

$$= \left[\sum_{n=2}^{\infty} \left(z^n \sum_{j=2}^{\infty} f_j \sum_{\substack{k_1+k_2,\dots,k_j=1 \\ k_1+\dots+k_j=n}}^{\infty} h_{k_1} \cdots h_{k_j} \right) \right]_{\text{term of degree } n}$$

$$= \sum_{j=2}^{n} f_j \sum_{\substack{k_1+\dots+k_j=n \\ k_i \geq 1}} h_{k_1} \cdots h_{k_j}$$

$\underline{n=2}$: $f_2 h_1^2$

$\underline{n=3}$: $f_2 \, 2h_1 h_2 + f_3 h_1^3$

$\underline{n=4}$: $f_2 (2h_1 h_3 + h_2^2) + f_3 (3 h_1^2 h_2) + f_4 h_1^4$

So we get the recursion

$$h_n = \frac{1}{\lambda^n - \lambda} \sum_{j=2}^{n} f_j \sum_{\substack{k_1+\dots+k_j=n \\ k_i \geq 1}} h_{k_1} \cdots h_{k_j}$$

theorem if $|\lambda| \neq 1$, then $h \in \mathbb{C}\{z\}$ (h converges)

$\lambda = e^{2\pi i \alpha}$; $\alpha \in \mathbb{R} \setminus \mathbb{Q}$; $\alpha \in DC(\tau)$

$$h_2 = \frac{f_2}{\lambda^2 - \lambda}$$

$$h_3 = \frac{1}{\lambda^3 - \lambda} \left(f_3 + 2 (\lambda^2 - \lambda)^{-1} f_2 \right)$$

$$h_4 = \frac{1}{\lambda^4 - \lambda} \Big(f_4 + 3 (\lambda^2 - \lambda)^{-4} f_3 f_2 + 2 (\lambda^3 - \lambda)^{-1} f_2 f_3$$
$$+ 4 (\lambda^3 - \lambda)^{-1} (\lambda^2 - \lambda)^{-1} f_2^3 + f_2^3 (\lambda^2 - \lambda)^{-2} \Big)$$

...

$$2^{n-2} \delta_2^{n-1} \left[(\lambda^n - \lambda) \dots (\lambda^3 - \lambda)(\lambda^2 - \lambda) \right]^{-1}$$

$$|\lambda^n - \lambda| = |\lambda^{n-1} - 1| \simeq 2 |\sin(n-1)\alpha| \approx 2(n-1)\alpha$$

$$\approx 2^{n-2} \delta_2^{n-1} (2\delta)^{(1+\tau)(n-1)} \left((n-1)!\right)^{1+\tau}$$

for $\left| \alpha - \frac{p}{q} \right| \geq \frac{\delta}{q^{2+\tau}} \Rightarrow |q\alpha - p| \geq \frac{\delta}{q^{1+\tau}}$

$$\Rightarrow \{ q\alpha \} \geq \frac{\delta}{q^{1+\tau}}$$

Proof of Koenigs - Poincaré:

$|\lambda^n - \lambda| \geq c > 0 \quad \forall n \in \mathbb{N}$

$\sigma_1 = 1$

$$\sigma_n = \sum_{2}^{n} \sum_{k_1 + \dots + k_j = n} \sigma_{k_1} \cdots \sigma_{k_j}$$

$$\sigma(z) = \sum_{1}^{\infty} \sigma_n z^n$$

$$\sigma(z) = z + \frac{(\sigma(z))^2}{1 - \sigma(z)}$$

$$\sigma(z) = \frac{1 + z - \sqrt{1 - 6z + z^2}}{4}$$