Sarig - Introduction to Ergodic Theory - 1

Basic problem: Understand the random behavior of deterministic dynamical systems.

Example (1) Coin ~~system~~ tossing

Example (2) Box filled w/ 1L of gas ($\sim 10^{24}$ molecules)

$\vec{q}_i$ = position $\vec{p}_i$ = momentum

$$\frac{d\vec{q}_i}{dt} = \frac{\partial H}{\partial \vec{p}_i}, \quad \frac{d\vec{p}_i}{dt} = -\frac{\partial H}{\partial \vec{q}_i}$$

Determinism: A continuous time dynamical system ("semiflow") is a 1-parameter family of maps $T_t : X \to X^{(t \geq 0)}$ s.t. $T_t \circ T_s = T_{t+s}$

Interpretation: • X is the "space", collection of all possible states of the system.

• T_t ("law of motion") takes state at time 0 and sends to state at time t.

• $T_t \circ T_s = T_{t+s}$ is a consistency requirement

Def (semi-orbit): Of X is $\{T_t(x)\}_{t=0}^{\infty}$

Def (measurement): Is a function $f : X \to \mathbb{R}$

Ex (2): $f(\vec{q}_1, \dots, \vec{q}_N, \vec{p}_1, \dots, \vec{p}_N) = \sum_{i=1}^{N} 1_{\left[\substack{\text{right}\\ \text{half of}\\ \text{the box}}\right]}(\vec{q}_i)$

Time series, $t \to f(T_t(x))$

Def: A discrete time dynamical system is $T : X \to X$.

Interpretation; $T : \begin{bmatrix}\text{state at}\\ \text{time } 0\end{bmatrix} \to \begin{bmatrix}\text{state at}\\ \text{time } 1\end{bmatrix}$.

$T^n : \begin{bmatrix}\text{state at}\\ \text{time } 0\end{bmatrix} \to \begin{bmatrix}\text{state at}\\ \text{time } n\end{bmatrix}$.

Mathematical setup for randomness

Example (3): Choose $x \in [0,1]$ randomly and keep it secret. I will answer a countable collection of "reasonable" questions on x.

Def: A probability space is $(\Omega, \mathcal{F}, \mu)$ where

(1) Ω is a set called "sample space" (ex(3): $\Omega = [0,1]$)

$\mathcal{F}$ is a σ-algebra:

(2) $\mathcal{F}$ is a collection of subsets of Ω, called measurable sets, s.t.: (ex(3): $\mathcal{F}$ = reasonable questions).

(a) $\emptyset \in \mathcal{F}$, $\Omega \in \mathcal{F}$

(b) If $E \in \Omega$, $E^c = \Omega \setminus E \in \mathcal{F}$

(c) If $E_i \in \mathcal{F}$ $\forall i \in \mathbb{N}$, then $\bigcup_{i=1}^{\infty} E_i, \bigcap_{i=1}^{\infty} E_i \in \mathcal{F}$.

(3) μ ("probability" measure) is a function $\mu : \mathcal{F} \to [0,1]$ s.t.

(a) $\mu(\emptyset) = 0$, $\mu(1) = 1$

(b) If $E_1, \dots, E_N, \dots \in \mathcal{F}$ are pairwise disjoint, then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

Theorem (Lebesgue): There exists a probability space $([0,1], \mathcal{B}, \mu)$ s.t.

(1) $\mathcal{B}$ contains all sub-intervals (and many more sets)

(2) $\mu([a,b]) = b - a$

(3) $\mu(a + E) = \mu(E)$ (translation invariance)

Theorem (Vitali): Lebesgue's Theorem is false for $\mathcal{F} = \{$ all subsets of $[0,1]\}$.

Def: A measurable function is $f : \Omega \to \mathbb{R}$ s.t. $\forall t$, $[f > t] := \{\omega \in \Omega : f(\omega) > t\}$ belongs to $\mathcal{F}$.

Exercise: f is measurable iff there is a countable list of reasonable question whose answers determines the value of f.

Def: A stochastic process is a sequence of measurable functions $f_n : \Omega \to \mathbb{R}$ on the same probability space.

$$\text{Prob}\left[a_k < f_{i_k}(\omega) < b_k : k \in [n]\right] = \mu\{\omega \in \Omega : a_k < f_{i_k}(\omega) < b_k\}$$

Ergodic Theory

Def: A probability preserving transformation is $(X, \mathcal{B}, T, \mu)$ where

(1) $(X, \mathcal{B}, \mu)$ is a probability space

(2) $T : X \to X$ is a measurable map, i.e., a map s.t $\forall E \in \mathcal{F}$, $T^{-1}(E) = \{x \in \Omega : T(x) \in E\} \in \mathcal{F}$

(3) $\mu(T^{-1}E) = \mu(E)$ $\forall E \in \mathcal{F}$.

Obs: Every measurement $f:\Omega \to \mathbb{R}$ gives rise to stochastic process
$f_i := f \circ T^i$.

Marmi – Introduction to small divisor problem - 1

X phase space (set + structure)

$End(X) = \{f : X \circlearrowleft \text{ preserving structure}\}$

Ex: $X=[0,1]$, $End(X) = \{\text{continuous } f\}$.

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Discrete time dynamical system: $f^m = \underbrace{f \circ \dots \circ f}_{m}$

$Aut(X) = \{f : X \to X \mid f^{-1} : X \to X\}$

$O_f(x) = \{f^j(x)\}_{j\in\mathbb{N}}$
(or $\mathbb{Z}$ case $f \in Aut(X)$)

Periodicity = finiteness of the orbit
$x_{m+1} = f(x_m)$: $x_0, x_1, \dots, x_{p-1}, x_p = x_0, x_{p+1} = x_1, \dots$

Quasiperiodicity: $0 < \varepsilon_m \to 0$ (error in the measure), $K_m \to \infty$
If $d(x,y) < \varepsilon_m$, they will be identified at step m.

Ex(1): $X = S^1 = \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$
$\mathbb{R} \to \mathbb{T}^1$
$x \mapsto [x] = \{x+p : p \in \mathbb{Z}\}$
$\alpha \in \mathbb{R}$, define $R_\alpha(x) = x+\alpha$ in $\mathbb{T}^1$.
$\alpha \in \mathbb{Q}$ then $\alpha = \frac{p}{q}$, $(p,q \in \mathbb{Z}) \Rightarrow$ all points are periodic w/ period p.
$\alpha \in \mathbb{R}\setminus\mathbb{Q}$, R_α is quasiperiodic
• $\{R_\alpha^j(x)\}_{j\in\mathbb{Z}}$ is dense in $\mathbb{T}^1$
"approximate periods" of R_α, R_α^q $q \in \mathbb{N}$ are such that
$|R_\alpha^q(x) - x| < \varepsilon$