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Lisbon, 2015
Course on Dynamical Systems

1.1 Introduction to Ergodic Theory

Problem

Understanding estocastic behaviour of deterministic dynamical systems.

Ex: Coin tossing, gas in a box (statistical mechanics).

1.1.1 Deterministic Setup. Definitions

It can be of continuous time system or discret time. If continuous time it is a 1-parametric family of maps $T_t : X \to X$ $(t \geq 0)$ / $T_t \circ T_s = T_{t+s}$

Where: X ("space") = {all possible states of the systems}

T_t ("law of motion"): $T_t : \binom{\text{initial}}{\text{state}} \longmapsto \binom{\text{state}}{\text{at time } t}$

$T_t \circ T_s = T_{t+s}$ (Consistency or translation time symetry)

(Semi) Orbit: $\{T_t(x)\}_t^\infty$ "Where the object is at each time" "en el espacio de posiciones y momentos"

Measurement: A function $f : X \to \mathbb{R}$
state $\mapsto$ Result of measurement)

In real life, we can't know the orbits, but we can know the results of the measurements, the set of wich ($\{f(T_t(x))\}_{t=0}^\infty$) is called time series

In a discret-time dynamical sistem is an only map $T: X \to X$ where $T: \left[\begin{smallmatrix}\text{state at}\\ t=n\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\text{state at}\\ \text{time } n+1\end{smallmatrix}\right]$.

~~Mes~~ And time series are analogous.

1.1.2. Randomness Setup. Definitions.

Example: Let us choose $x \in [0,1]$, at randomly. We'll keep it secret, but we will answer any countable collection of "reasonable" questions.

Probability space: $(\Omega, \mathcal{F}, \mu)$

(1): Ω "sample space"

(2): $\mathcal{F}$ ("σ-algebra") is a collection of subsets of Ω called "measurable sets") /

- $\mathcal{F}$ contains Ω and $\emptyset$
- If $\mathcal{F} \ni E$, then E^c in Ω, $E^c \in \mathcal{F}$ (Closed under complementarity)
- If $E_i \in \mathcal{F}$ $\forall i \in \mathbb{N}$, then $\bigcup_{i=1}^\infty E_i$, $\bigcap_{i=1}^\infty$ belong to $\mathcal{F}$ (countable !!! unions and intersections)

(3): μ: Probability measure": function / $\mu : \mathcal{F} \to [0,1]$ /

- $\mu(\emptyset) = 0$; $\mu(\Omega) = 1$
- If $\{E_i\}_{i=1}^\infty$ is a countable set of pairwise disjoint sets:

$$\mu\left(\coprod_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

Thm. (Lebesgue) There exists a probability space $([0,1], \beta, \mu)$ where:

(1) β (álgebra de Borel) contains all subintervals, [but not all subsets!!!]

(2) $\mu([a,b]) = b-a$

(3) μ (Invariant under translations)

Measurable function $f: \Omega \to \mathbb{R} / \ \forall t \ [\underbrace{f > t}_{\text{imagen inversa de } (t,\infty)}] := \{\omega \in \Omega \mid f(\omega) > t\}$

A demostrar: f is measurable iff there is a countable list of measurable questions whose answer is binary and determine f

Stochastic process is a sequence of measurable function $f_n: \Omega \to \mathbb{R}$ on the same probability space.

$\text{Prob}\,[a_k < f_k(\omega) < b_k] \quad (k = 1, \dots, n) \equiv \mu\{\omega \in \Omega \mid f_k(\omega) \mid a_k \ \text{and} \ f_k(\omega) < b_k\}_{k=1,\dots,n}$

11.3. Ergodic theory

A probability preserving transformation is (X, β, T, μ) where

(1) (X, β, μ) is a probability space

(2) $T: X \to X$ is a measurable map i.e. a map. /
$\forall E \in \mathcal{F} ; T^{-1}(E) = \{\omega \in \Omega \mid T(\omega) \in E\}$ is in $\mathcal{F}$

(3) $\mu(T^{-1}E) = \mu(E) \quad \forall E \in \mathcal{F}$