Series - 4

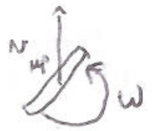
Independence and Mixing:

Probability Theory: Suppose $(\Omega, \mathcal{F}, \mu)$ is a prob. space. Two events are independent if $\mu(E \cap F) = \mu(E) \cdot \mu(F)$.

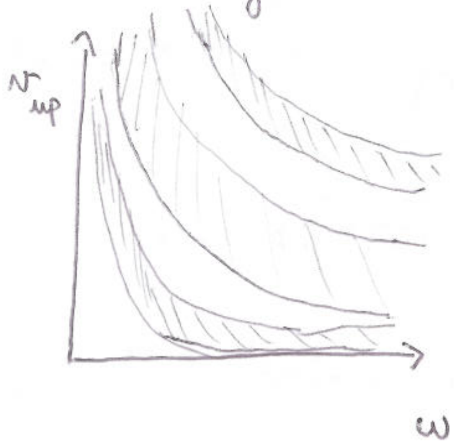
In this case $Pr(E|F) = \dfrac{\mu(E \cap F)}{\mu(F)} = \mu(E)$ and $Pr(F|E) = \mu(F) = Pr(F)$

Approximate independence: $\mu(E \cap F) \approx \mu(E)\mu(F)$

Ex: Coin Toss



$t_{\text{in the air}} = \dfrac{2 v_{up}}{g}$, N = number of turns $= \left[\dfrac{2 v_{up} \omega}{g \cdot \pi}\right]$. When is this odd?



$v_{up} \cdot \omega = \dfrac{g k}{2}$, $k \in \mathbb{Z}^+$.

Dynamical system

Def: A probability preserving system $(\Omega, \mathcal{F}, \mu, T)$ is called mixing if $\forall E, F \in \mathcal{F}$,

$$\mu(E \cap T^{-n} F) \xrightarrow[n \to \infty]{} \mu(E)\mu(F) \quad \rightarrow \text{"Asymptotic unpredictability"}$$

That is, $Pr(T^n(\omega) \in F \mid \omega \in E) = Pr(\omega \in F)$.

Proposition: A probability preserving system $(\Omega, \mathcal{F}, \mu, T)$ is ergodic iff $\forall E, F \in \mathcal{F}$,

$$\frac{1}{N}\left(\sum_{k=0}^{N-1} \mu(E \cap T^{-k} F)\right) \longrightarrow \mu(E)\mu(F).$$

In particular mixing $\Rightarrow$ Ergodic.

Proof: Suppose T is ergodic and $E, F \in \mathcal{F}$. Ergodic Thm gives

$$\frac{1}{N}\left(\sum_{k=0}^{N-1} 1_F \circ T^k\right) \longmapsto \int 1_F \, d\mu = \mu(F)$$

So $$\mu(F)\, 1_E(\omega) = \lim_{N\to\infty} \frac{1}{N}\left(\sum_{k=0}^{N-1} 1_F(T^k\omega)\, 1_E(\omega)\right) \text{ a.e.}$$

$$= \lim_{N\to\infty} \frac{1}{N}\left(\sum_{k=0}^{N-1} 1_{E\cap T^{-k}F}(\omega)\right)$$

$$\Rightarrow \int_\Omega \mu(F)\cdot 1_E(\omega)\, d\mu = \int \lim_{N\to\infty} \frac{1}{N}\left(\sum_{k=0}^{N-1} 1_{E\cap T^{-k}F}(\omega)\right)$$

|| bounded convergence

$$\mu(F)\mu(E) \qquad \lim_{N\to\infty} \frac{1}{N} \int \sum_{k=0}^{N-1} 1_{E\cap T^{-k}F}(\omega)$$

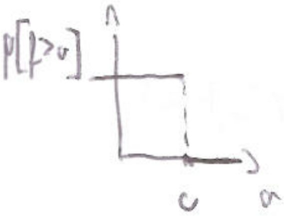
$$= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mu(E\cap T^{-k}F)$$

($\Leftarrow$) Suppose f is an invariant function, that is, $f = f\circ T$ a.e. and limit holds. Then $T^{-1}[f>a] = [f>a]$.

Take $E = F = [f>a]$.

Then $$\frac{1}{N}\sum_{k=0}^{N-1} \cancel{\mu[f>a]} \underbrace{\mu([f>a]\cap T^{-k}[f>a])}_{\mu[f>a]} = \mu[f>a]^2$$

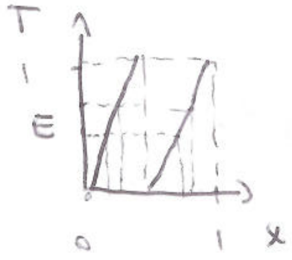
$$\Rightarrow \mu[f>a] = \mu[f>a]^2 \Rightarrow \mu[f>a] = \begin{cases}0\\1\end{cases}$$



Then $f = c$ a.e. $\square$

Example (Angle Doubling Map): $(\Omega, \mathcal{F}, \mu, T)$, $\Omega = [0,1]$, $\mathcal{F}$ = Borel σ-Algebra $= \bigcap_{\mathcal{E}} \{\mathcal{E} \subseteq 2^{\Omega} \mid \mathcal{E}$ is a σ-algebra and $\mathcal{E} \supseteq \{\text{intervals}\}\}$, μ = Lebesgue Measure,

$T(x) = 2x \mod 1$



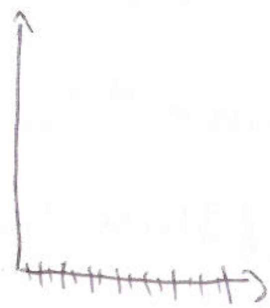
$\mu(T^{-1}E) = \frac{1}{2}\mu(E) + \frac{1}{2}\mu(E) = \mu(E)$ for intervals. Therefore collection of sets s.t. $\mu(T^{-1}E) = \mu(E)$ is a σ-algebra and contains intervals $\Rightarrow$ it contains Borel σ-algebra.

If $x = 0.x_1x_2x_3\ldots$ (base 2), then $T(x) = 0.x_2x_3\ldots$ Therefore

$T^k(x) = 0.x_{k+1}x_{k+2}\ldots$

Dyadic intervals: $[a_1, \ldots, a_m] = \{x : x = 0.a_1a_2\ldots a_m k_{m+1}\ldots ; k_i \in \{0,1\}\}$

$$= \left[\sum_{k=1}^{m} \frac{a_k}{2^k}, \sum_{k=1}^{m} \frac{a_k}{2^k} + 2^{-m}\right]$$



For any 2 dyadic ~~(systems)~~ intervals

$[a_0, a_1, \ldots, a_{\alpha-1}]$, $[b_0, \ldots, b_{\beta-1}]$ we have

$$[\underline{a}] \cap T^{-m}[\underline{b}] = [a_0, \ldots, a_{\alpha-1}, x, \ldots, x, b_0, \ldots, b_{\beta-1}] \quad \text{if } m > \alpha$$

$$= \underbrace{\biguplus_{x_\alpha, \ldots, x_{m-1} = 0,1} [\underline{a}, \underline{x}, \underline{b}]}_{2^{m-\alpha} \text{ intervals of length } 2^{m+\beta}}$$

Therefore $\lambda([\underline{a}] \cap T^{-m}[\underline{b}]) = 2^{m-\alpha} \cdot 2^{-m-\beta} = \frac{1}{2^\alpha} \frac{1}{2^\beta} = \lambda([\underline{a}])\lambda([\underline{b}])$ $\forall m > \alpha$.

Approx lemma: For each Borel set E and $\varepsilon > 0$ there are dyadic intervals $I_1, \dots, I_m$ such that
$$\mu\left(E \,\Delta\, \bigcup_{i=1}^{m} I_i\right) < \varepsilon.$$

Bergelson-2

If $\bar{d}(E) = \overline{\lim} \frac{|E \cap [n]|}{n} > 0$, $\exists m \in \mathbb{N}$ s.t. $\bar{d}(E \cap (E - m)) > 0$.

we have $\bar{d}(E) = \bar{d}(E - k)$.

$\bar{d}(E) > 0 \Leftrightarrow$ For some sequence N_k, $\frac{|E \cap [N_k]|}{N_k} \xrightarrow[k \to \infty]{} \bar{d}(E) > 0$

In Poincaré recurrence theorem the first $m > 0$ such that $\mu(A \cap T^{-m}A) > 0$ is bounded by $\frac{1}{\mu(A)}$.

Define density in $\mathbb{N} \times \mathbb{N}$ as $\bar{d}(A) = \overline{\lim_{N \to \infty}} \frac{|A \cap [N]^2|}{N^2}$.

Exercise: If $\bar{d}(A) > 0$, then for many (k, l) we have $\bar{d}(A \cap (A - (k, l))) > 0$

Example: $d(2\mathbb{N}+1 \times 17\mathbb{N}) = \frac{1}{34}$, $d(\{(a, b) : a \perp b\}) = \frac{6}{\pi^2}$.

Results: 1 Poincaré: $\forall (X, \mathcal{B}, \mu, T)$ $\exists m < \frac{1}{\mu(A)}$: $\mu(A \cap T^{-m}A) > 0$.

2 Corollary of ergodic theorem: $\cancel{f} = \mathbb{1}_A$ mixing with $A = B$
$$\frac{1}{N} \sum_{m=0}^{N-1} \mu(A \cap T^{-m}A) \xrightarrow[n \to \infty]{} \mu^2(A)$$

Theorem (Khintchine):
For any T (not necessarily ergodic), $\lim_{N \to \infty} \frac{1}{N} \sum_{m=0}^{N-1} \mu(A \cap T^{-m}A) \geq \mu^2(A)$.

Def: A set $S \subseteq \mathbb{N}$ is called syndetic if it has bounded gaps
(i.e. $\exists m_1, \dots, m_k$: $\bigcup_{i=1}^{k} (S - m_i) = \mathbb{N}$).