10/7/2015

$\boxed{1^{st} \text{ class}}$

$(X, \mathcal{B}, \mu, T)$

Every $f: X \to \mathbb{R}$ generates a stochastic process $\{f \circ T^m\}_{m \in \mathbb{N}}$

Question: "How random" can a system lost

Entropy $\{X_m\}_{m=1}^{\infty}$

~~Bernoulli~~

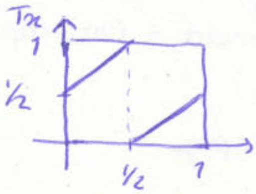
Definition: A stochastic process is called Bernoulli if $\exists$ finite set $\{a_1, \dots, a_N\}$ and a vector $\vec{p} = (p_1, p_2, \dots, p_N)$, $0 \le p_i \le 1$, $\sum p_i = 1$, s.t.

(1) $\mathrm{Prob}(X_i = a_k) = p_k$ for all i.

(2) X_i are independent : for all $a_{i_1}, \dots, a_{i_m}$ $\Pr[X_{m} = a_{i_m}, X_{m+1} = a_{i_{m+1}}, \dots, X_{n} = a_{i_n}] = p_{i_m} p_{i_{m+1}} \cdots p_{i_n}$.

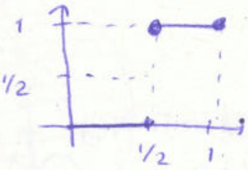
Definition: ~~We s~~ A probability preserving system $(\Omega, \mathcal{B}, \mu, T)$ simulates a Bernoulli process, if $\exists$ a measurable function $f: \Omega \to \{a_1, \dots, a_N\}$ s.t. $\{f \circ T^i\}_{i=1}^{\infty}$ is a Bernoulli process i.e. $\exists$ prob. vector $(p_1, \dots, p_N)$ s.t.

$$\mu\{\omega \in \Omega \mid f(T^k \omega) = a_{i_k},\ k = 0, \dots, m-1\} = \prod_{k=0}^{m-1} p_{i_k}$$



Example (one-sided): $T(x) = 2x \pmod 1$

$$f(x) = \begin{cases} 0, & [0, 1/2) \\ 1, & [1/2, 1] \end{cases}$$



Suppose $x = 0.x_1 x_2 x_3 \dots$ (binary)

Then $T^i(x) = 0.x_{i+1} x_{i+2} x_{i+3} \dots$ (binary)

whence $f(T^i(x)) = x_{i+1}$.

$$\mu\{x \in [0,1] : f(x) = a_1, f(Tx) = a_2, \dots, f(T^m x) = a_{m+1}\}$$

$$= \mu\{x = 0.a_1 a_2 \dots a_{m+1} \ast\ast\ast\} = \mu\left[\sum_{k=0}^{m+1} \frac{a_k}{2^{k+1}}, \sum_{k=0}^{m+1} \frac{a_k}{2^{k+1}} + \frac{1}{2^{m+1}}\right] = \left(\frac{1}{2}\right)^{m+1}$$

Definition: the entropy of a Bernoulli process with prob. vector $(p_1, \dots, p_N)$ is the (non-negative) number $-\sum_{i=1}^{N} p_i \log p_i$.

Theorem: (Sinai): If a probability preserving system simulates a Bernoulli process with entropy H, then it simulates any Bernoulli process with entropy smaller than H.

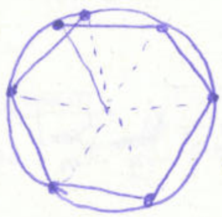
- The bigger the entropy H, more random will the system be.

<u>Definition</u>: The <u>metric entropy</u> of a probability preserving system is $\sup\{H \mid$ the system simulates a Bernoulli-process with entropy $H\}$.
(this is a maximum).

SYSTEMS WITH POSITIVE ENTROPY.

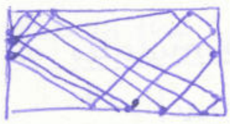
Example 1: Geodesic flow on compact surfaces with negative curvature.

Example 2:



It has zero entropy.

the ROTATION



zero entropy, aswell.

positive entropy !!

Example 3: (Arnold's Cat Map): $T: \mathbb{T}^2 \to \mathbb{T}^2$ $\quad T(x,y) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ mod $\mathbb{Z}^2$ ~~and $\mathbb{Z}^2$~~

↳ positive entropy

→ which mechanisms produce positive entropy?

<u>Setup</u>: Ω is a compact smooth manifold

and $T: \Omega \to \Omega$ is a twice continuously differentiable invertible map.

$\exists \varepsilon_0 > 0$

<u>Definition</u>: T has <u>exponential sensitivity</u> to initial conditions at $x \in \Omega$ if $\exists \varepsilon_0 > 0$ $\exists \lambda > 0$ s.t. for every $n > 0$ $\exists y \in \Omega$ s.t.
(1) $d(y,x) < e^{-\lambda n}$
(2) $d(T^n(y), T^n(x)) > \varepsilon_0$.

exponentially close $\begin{cases} \cdot\, y & \cdot\, Ty & \dots & \cdot\, T^n_y \\ \cdot\, x & \cdot\, Tx & \dots & \cdot\, T^n_x \end{cases}$ noticeable error

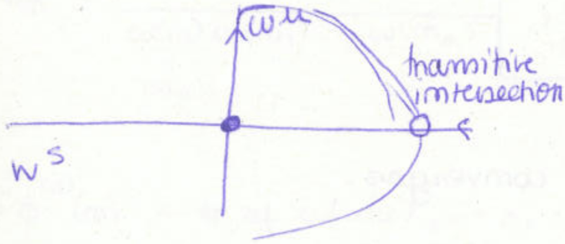
Theorem (Ruelle Margulis): Under the two previous conditions, if T preserves an ergodic problem measure with positive entropy, then μ-a.e. x has exponential sensitivity to initial condition.

⚠ there must be exponential sensitivity to initial conditions in order to have a stochastic random behaviour.

From now on, assume T is a twice continuous differential ~~equation~~ invertible map on a compacted ~~manyfold~~ manifold of dim. 2.

Theorem (Katok, 1980): If T has ergodic invariable problem measure with positive entropy, then the following picture must appear somewhere in the manifold.

There is a periodic point P



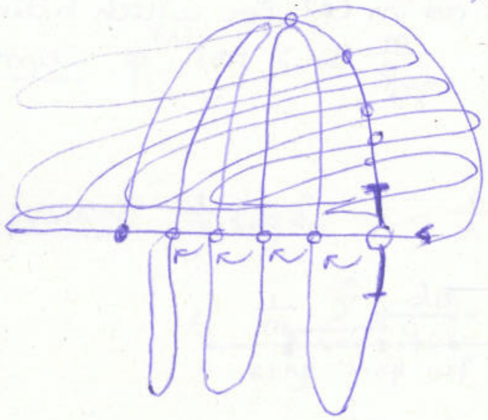
$T(W^s) \subseteq \cancel{T}(W^s)$

$T|_{W^s}$ contracts uniformely.

$T^{-1}(W^u) \subseteq \cancel{T}(W^u)$

$T^{-1}|_{W^u}$ contracts uniformely

$\Leftrightarrow T|_{W^u}$ expands.



(?)

KATOK HORS~~ES~~HOE

Example:

$T(x) = 2x \pmod 1$ $\quad h = \log 2$

$T(x,y) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \bmod \mathbb{Z}^2$ $\quad \log|\text{larges e.v.}|$

Geodesic flow [constant neg. curve] $h(g^t) = |t|$