1.2. Basic Ergodic theory

We are interested in the beaviour of a system's time series.

<u>Strong law of Large Numbers</u>

"Let X_i be an independent, identically distributed set of random variables. Then, the average converges to the expected value almost always"

Also: $\frac{1}{n}\sum^{n} X_i \to |E(X_1)|$; $\frac{1}{n}\sum_{k=0}^{n-1} f(T^k\omega) \longrightarrow \int_\Omega f\, d\mu$

<u>Recall</u>: Measurable functions & there integrals

Remember: $f:\Omega\to\mathbb{R}$ is measurable iff $\forall a\in\mathbb{R}$ $[f>a]$ is in $\mathcal{F}$ (σ-algebra of Ω) (Mind we are using β algebra in the reals, generated by $\{[a,\infty)\}_{a\in\mathbb{R}}$

Examples of measurable functions are indicators of measurable sets, or:

- "Simple functions" = "discrete random variable"

  $$f=\sum_{i=1}^{N}\alpha_i \mathbb{1}_{E_i} \quad (E_i\in\mathcal{F}\ \forall i\in\{1,\dots,N\})$$

- <u>Pointwise limits of simple function.</u>

  Because: If $f_n:\Omega\to\mathbb{R}$ is measurable and $|f_n|\le M$ $\forall n\in\mathbb{N}$, $\limsup_{n\to\infty} f_n(\omega)$, $\liminf_{n\to\infty} f_n(\omega)$ is measurable $\forall n\in\mathbb{N}$

  <u>Proof</u>: $\limsup_{n\to\infty} f_n(\omega)>a \Leftrightarrow \exists n \,/\, (\limsup_{k\to\infty} f_k(\omega)\ge a+\frac{1}{n}) \Leftrightarrow$

  $\exists n \,/\, \forall m\ \forall N\ \exists k>N\ (f_k(\omega)>a+\frac{1}{n}+\frac{1}{m}) \Leftrightarrow [\limsup f>a]=\bigcup_{n=1}^{\infty}\bigcap_{m=1}^{\infty}\bigcap_{N=1}^{\infty}\bigcup_{k=N+1}^{\infty}[f_k>a+\frac{1}{m}+\frac{1}{n}]$

And since σ-algebras are closed under numerable unions and intersections, $[\limsup f>a]\in\mathcal{F}$. ∎

Note: This doesn't hold for continuity. This is a standard form of reasoning when working wit σ-algebras: translate to countable conditions and expressing them as set unions and intersections.

This is deeper, as:

<u>Proposition</u> Every bounded measurable f is the uniform limit of a sequence of simple f_n

<u>Proof</u> Suppose $|f|\le M$, f measurable. Let: $f_n := \sum_{k=-M_n}^{M_n}\frac{k}{n}\mathbb{1}_{[\frac{k}{n}\le f\le\frac{k+1}{n}]}$

Then, for every $\omega\in\Omega$, there is a <u>unique</u> $k \,/\, \frac{k}{n}<f(\omega)\le\frac{k+1}{n} \Leftrightarrow f(\omega)\in(\frac{k}{n},\frac{k+1}{n})$

So $|f_n(\omega)-f(\omega)|\le\frac{1}{n}\xrightarrow[n\to\infty]{}0$ ∎

Now, lets see how we can integrate that.

- Indicators: Clearly: $\int_\Omega \mathbb{1}_E := \mu(E)$ is the most reasonable definition.
- Simple functions: $f=\sum_i\alpha_i\mathbb{1}_{E_i} \Rightarrow \int_\Omega f\,d\mu \overset{\text{definition + linearity}}{=} \sum_i\alpha_i\mu(E_i)$

- For general bounded measurable function: since $f = \lim_{n\to\infty} f_n$ (f_n simple)

So we define: $\int_\Omega f \, d\mu := \lim_{n\to\infty} \int_\Omega f_n \, d\mu$.

We can show that this limit exists and $\int$ has the usual properties of integrals.

Proof: (Existence of the limit)

Using the Cauchy criterion. Let $\varepsilon > 0$. There is $n_0 \in \mathbb{N} / |f_n(\omega) - f(\omega)| < \varepsilon \ \forall \omega$.

Then $\forall n, m \geq n_0 : |f_n(\omega) - f_m(\omega)| \leq |f_n(\omega) - f(\omega)| + |f(\omega) - f_m(\omega)| < 2\varepsilon$

By the triangle inequality: $|\int f_n d\mu - \int f_m d\mu| \leq \int |f_n - f_m| d\mu \leq \int 2\varepsilon \, d\mu = 2\varepsilon$

So $\int f_n$ is a Cauchy sequence in $\mathbb{R}$, wich is complete, so it's convergent.

The usual properties are left to check.

Proposition If $\{f_n\}, \{g_n\}$ sequences of simple functions that converge to f, then

$$\lim_{n\to\infty} \int_\Omega f \, d\mu = \int_\Omega g_n \, d\mu$$

Proof Fix $\varepsilon > 0$. Choose $n_0 \in \mathbb{N} / \forall n \geq n_0 \begin{cases} |f_n(\omega) - f(\omega)| < \varepsilon \\ |g_n(\omega) - f(\omega)| < \varepsilon \end{cases} \quad \forall \omega \in \mathcal{F}$

Then, necessarily $|f_n(\omega) - g_n(\omega)| < 2\varepsilon \Rightarrow$

$|\int f_n d\mu - \int g_n d\mu| \leq \int |f_n - g_n| d\mu \leq 2\varepsilon \Rightarrow \lim \int f_n = \lim \int g_n$ ∎

Point-wise Ergodic theorem

We want to prove the law of large number: $\frac{1}{N} \sum_{k=0}^{N-1} f(T^k \omega) \xrightarrow[N\to\infty]{} \int f \, d\mu$ almost everywhere.

Let us note that in the case of conserved quantities (invariant functions) we have an obstruction.

Def. $f : \Omega \to \mathbb{R}$ is called T-invariant, if $f \circ T = f$ almost everywhere

A probability preserving system $(\Omega, \mathcal{B}, \mu, T)$ is ergodic iff every T-invariant function $f : \Omega \to \mathbb{R}$ is equal to a constant function a.e.: $\exists c / \mu[f \neq c] = 0$.

We shall now precisely state Birkhoff Pointwise Ergodic Thm:

"Suppose $(\Omega, \mathcal{B}, \mu, T)$ measure preserving system, and $f : \Omega \to \mathbb{R}$ be a bounded measurable function. Then:

$$\bar{f}(\omega) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f(T^k(\omega)) \text{ exists a.e.}$$

If $(\Omega, \mathcal{B}, \mu, T)$ is ergodic, then $\bar{f}(\omega) = \int_\Omega f \, d\mu$ alm. ev."

Proof Let's first assume $|f| \leq 1$; $f \in [0,1]$.

Let $\bar{A}(\omega) := \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N-1} f(T^k \omega)$

$\underline{A}(\omega) := \liminf_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N-1} f(T^k \omega)$

We'll show $\bar{A}(\omega) = \underline{A}(\omega)$ a.e. Since $\bar{A} \geq \underline{A}$: $\bar{A} = \underline{A}$ a.e $\Leftrightarrow \mu[\bar{A} > \underline{A}] = 0 \Leftrightarrow$

$\forall n \ \mu[\bar{A} - \underline{A} > \frac{1}{n}] = 0$

If we could show that $\int (\bar{A}-\underline{A})\,d\mu = 0$, this will imply: $\mu[\bar{A}-\underline{A}\geq \frac{1}{n}] \leq \int 1_{[\bar{A}-\underline{A}\geq \frac{1}{n}]}\,d\mu \leq$
$\leq \int n\cdot(\bar{A}-\underline{A})\,d\mu = 0$. and thus $\bar{A}=\underline{A}$ a.e.

We start proving $\int(\bar{A}-\underline{A})\,d\mu=0$. We fix $M>0$, $\varepsilon>0$.

We define $\tau(\omega) = \min\{n>0 \mid \frac{1}{n}\sum_{k=0}^{n-1} f(T^k\omega) > \bar{A}(\omega)-\varepsilon\}$ (well defined)

We colour the time intervals between integers:

(1) If $\tau(\omega) > M$, k is "red"

(2) If not, it is blue.

And then, we move to the next colorless interval.

~~and~~ you stop when all intervals are colored or when you want to color a segment blue, but there are not enough k's left.

We estimate

$$\sum_{k=0}^{N-1} (f(T^k(\omega)) + 1_{[\tau\geq M]}(T^k\omega)) \text{ from below}$$

If k is red, $T^k\omega \in [\tau>M]$: $\sum_{k \text{ red}} (\underbrace{f(T^k\omega)}_{\geq 0} + \underbrace{1_{[\tau\geq M]}(T^k\omega)}_{\geq 1}) \geq (\#\text{reds})\cdot(\bar{A}(\omega)-\varepsilon)$

If k is blue $\sum_{k \text{ blue}} (\sum_{k \text{ segments}} (f(T^k\omega) + 1_{[\tau\geq M]}(T^k\omega))) \geq \sum_{k \text{ blue}} (\sum_{\text{segment}} f(T^k\omega)) \geq \sum_{k \text{ blue}} (\text{length})\cdot (\bar{A}(T^{k_{\text{begin}}}\omega)-\varepsilon) =$

$= (\text{length})\cdot(\text{average}) \geq (\#\text{blues})\cdot(\bar{A}(\omega)-\varepsilon)$

So:

$$\sum_{k=0}^{N-1} (f(T^k\omega) + 1_{[\tau>M]}(T^k\omega)) \geq \sum_{\text{red}} + \sum_{\text{blue}} + \overbrace{\sum_{\text{colorless}}}^{>0} \geq (\#\text{reds} + \#\text{blues})(\bar{A}(\omega)-\varepsilon) \geq$$
$$\geq (N-M)(\bar{A}(\omega)-\varepsilon)$$

less than M are colorless

Thus $\frac{1}{N}\sum_{k=0}^{N-1} (f(T^k\omega) + 1_{[\tau>M]}(T^k\omega)) \geq (1-\frac{M}{N})(\bar{A}(\omega)-\varepsilon) \geq (1-\frac{M}{N})\int_\Omega \bar{A}-\varepsilon$ and

since μ is T-invariant

Exercise: $\int g\circ T^k = \int g$ for all g bdd measurable when μ is T-invariant

$$\frac{1}{N}\sum_{k=0}^{N-1} \left(\int_\Omega f\,d\mu + \mu[\tau>M]\right) \geq \left(1-\frac{M}{N}\right)\left(\int_\Omega \bar{A}\,d\mu - \varepsilon\right) \Rightarrow$$

$\int_\Omega f\,d\mu + \mu[\tau>M] \geq$

Taking $N\to\infty$, $\varepsilon\to 0$, $M\to\infty$ gives $\boxed{\int_\Omega f\,d\mu \geq \int \bar{A}\,d\mu - \cancel{\varepsilon}^{\,0}}$ ∎

It is easy to see that $\bar{f}(\omega) = \lim \frac{1}{N}\sum_{k=0}^{N-1} f(T^k\omega)$ is invariant ($\bar{f}\circ T = \bar{f}$)
So, in the ergodic case $\exists c / \bar{f}(\omega) = c$ a.e.