(X, B, μ, T) ; $\mu(T^{-1}E) = \mu(E)$

$f : \Omega \to \mathbb{R}$

$f(T^k(\omega))$

Strong Law of Large Variables

X_i independent dist. r.v.

$$\frac{X_1 + \dots + X_n}{n} \longrightarrow \mathbb{E}(X_0)$$

prob 1

$$\frac{1}{n} \sum_{k=0}^{n-1} f(T^k \omega) \xrightarrow[n \to \infty]{} \int_\Omega f \, d\mu \quad \text{almost everywhere}$$

Measurable functions and their integrals

Recall: $f : \Omega \to \mathbb{R}$ is measurable (w.r.t. a σ-algebra $\mathcal{F}$) if $\forall a \in \mathbb{R}$, $[f > a] := \{\omega \in \Omega \mid f(\omega) > a\}$ is in $\mathcal{F}$.

Example 1: Indicators of measurable sets

$$1_E(\omega) := \begin{cases} 1, & \omega \in E \\ 0, & \omega \notin E \end{cases} \qquad (E \in \mathcal{F})$$

$$[1_E > a) = \begin{cases} \Omega, & \text{se } a < 0 \\ E, & \text{if } 0 \leq a < 1 \\ \emptyset, & \text{if } a \geq 1 \end{cases}$$

$1_E(\omega)$ is measurable

Example 2 ("simple functions" / "discrete random variables")

$$f = \sum_{i=1}^{N} \alpha_i 1_{E_i} \qquad (E_1, \dots, E_n \in \mathcal{F})$$

Example 3

Pointwise limits of simple functions

This follows from:

Prop: If $f_n : \Omega \to \mathbb{R}$ are measurable and $|f_n| \leq M$, then $\limsup_{n\to\infty} f_n(w)$, $\liminf_{n\to\infty} f_n(w)$ are measurable.

[Recall: If $(a_n)_{n\geq 0}$ is bounded, then

$$\limsup_{n\to\infty} a_n = \max \left\{ L : \begin{array}{c} \exists \text{ subsequence} \\ a_{n_k} \to L \end{array} \right\}$$

$$\liminf_{n\to\infty} a_n = \min \left\{ L : \begin{array}{c} \exists \text{ subsequence} \\ a_{n_k} \to L \end{array} \right\}$$]

Proof of the prop:

$$\limsup_{n\to\infty} f_n(w) > a \iff \exists n \left(\limsup_{m\to\infty} f_m(w) > a + \frac{1}{n} \right)$$

$$\iff \exists n \left(\forall m \, \forall N \, \exists k > N \left(f_k(w) > a + \frac{1}{n} - \frac{1}{m} \right) \right)$$

therefore $$[\limsup > a] = \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{k=N+1}^{\infty} \left[f_k > a + \frac{1}{n} - \frac{1}{m} \right]$$

$$\in \mathcal{F}$$

Prop 2: Every bounded measurable function is the uniform limit of some simple function

Proof: Suppose $|f| \leq M$, f measurable.

Let $$f_n = \sum_{k=-Mn}^{Mn} \frac{k}{n} 1_{\left[\frac{k}{n} < f \leq \frac{k+1}{n}\right]}$$

For every $\omega \in \Omega$, there's a unique k s.t. $\frac{k}{n} < f(\omega) \leq \frac{k+1}{n}$

For this k, $f(\omega) \in \left(\frac{k}{n}, \frac{k+1}{n}\right]$, $f_n(\omega) = \frac{k}{n}$

so $|f_n(\omega) - f(\omega)| < \frac{1}{n}$

$$\Rightarrow \sup |f_n - f| < \frac{1}{n} \xrightarrow[n \to \infty]{} 0 \qquad \square$$

Definition of the integral

Case 1: Indicators for $f = 1_E$, $E \in \mathcal{F}$.

$$\int_\Omega 1_E \, d\mu := \mu(E)$$

Case 2: Simple functions: $f = \sum_{i=1}^{N} \alpha_i 1_{E_i}$

$$\int_\Omega f \, d\mu := \sum_{i=1}^{N} \alpha_i \mu(E_i)$$

Suppose $E_1, \dots, E_N$ are disjoint

Then $\mathbb{P}\mathrm{rob}[f = \alpha_i] = \mu(E_i)$

So $$\int_\Omega f \, d\mu = \sum_{i=1}^{N} \alpha_i \, \mathbb{P}\mathrm{rob}[f = \alpha_i] = \mathbb{E}(f)$$

Case 3 : General bounded measurable functions.

Every function f like this is the uniform limit of simple functions. $f = \lim_{n\to\infty} f_n$, f_n simple

Define $$\int_\Omega f \, d\mu := \lim_{n\to\infty} \int_\Omega f_n \, d\mu .$$

Proposition : The limit exists, is independent of the choices of f_n, and has the following properties.

(1) Linearity : $\int_\Omega (\alpha f + \beta g) \, d\mu = \alpha \int_\Omega f \, d\mu + \beta \int_\Omega g \, d\mu$

(2) Positivity : $f \geq 0 \Rightarrow \int_\Omega f \, d\mu \geq 0$

(3) Monotinicity : $f \geq g \Rightarrow \int_\Omega f \, d\mu \geq \int_\Omega g \, d\mu$

(4) Δ-inequality : $\left| \int_\Omega f \, d\mu \right| \leq \int_\Omega |f| \, d\mu$

Proof : For simple functions : exercise

Suppose f is bounded measurable. Let f_n b simple functions such that $\sup |f_n - f| \xrightarrow[n \to \infty]{} 0$

<u>Claim</u>: $\lim \int_\Omega f_n \, d\mu$ exists

<u>Proof</u>: We check the Cauchy criterion. Fix $\varepsilon > 0$.

Choose N s.t. $n > N \Rightarrow |f_n(w) - f(w)| < \varepsilon$ for all w.

Then for all $m, n > N$ $\quad |f_n(w) - f_m(w)| \leq |f_n(w) - f(w) + (f(w) - f_m(w))|$

$< 2\varepsilon \Rightarrow$ Cauchy ~~[illegible]~~

By ~~the~~ Δ-inequality and linearity (for simple functions):

$$\left| \int f_n \, d\mu - \int f_m \, d\mu \right| = \left| \int (f_n - f_m) \, d\mu \right| \leq \int |f_n - f_m| \, d\mu$$

$$\leq \int (2\varepsilon) \, d\mu = 2\varepsilon$$

So $a_n = \int f_n d\mu$ is a Cauchy sequence //

<u>Claim</u>: If $\{f_n\}$, $\{g_m\}$ are two sequences of simple functions which converge uniformly to f, then

$$\lim_{n \to \infty} \int_\Omega f_n \, d\mu = \lim_{m \to \infty} \int_\Omega g_m \, d\mu$$

<u>Proof</u>: Fix $\varepsilon > 0$. Choose N s.t.

$$n > N \Rightarrow \begin{cases} |f_n(w) - f(w)| < \varepsilon \\ |g_n(w) - f(w)| < \varepsilon \end{cases} \quad \text{for all } w$$

Necessarily $|f_n(w) - g_m(w)| < 2\varepsilon$ for $n > N$.

and we will have $\left| \int f_n - \int g_m \right| \leq 2\varepsilon \Rightarrow$ Cauchy $\Rightarrow$

Ex

$$\mathbb{D}(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & cc \end{cases}$$

$$\int_{[0,1]} \mathbb{D}\, dx = 1\, \mu(\mathbb{Q}) + 0\, \mu(\mathbb{Q}^c)$$

$$= 1 \times 0 + 0 \times 1 = 0$$

Bounded Convergence theorem

Suppose f_n are measurable functions on a probability space s.t.

$f(\omega) = \lim_{n\to\infty} f_n(\omega)$ exists for all ω

if $\exists M > 0$ s.t. $|f_n| \le M$ for all n, then

$$\int_\Omega \lim_{n\to\infty} f_n\, d\mu = \lim_{n\to\infty} \int_\Omega f_n\, d\mu$$

The Ergodic Theory / Strong LLN

We want: $\frac{1}{N} \sum_{k=0}^{N-1} f(T^k \omega) \xrightarrow[n\to\infty]{} \int f\, d\mu$ a.e.

Obstruction: Conservation laws

Def: A measurable function $f : \Omega \to \mathbb{R}$ is called T-invariant if

$f \circ T = f$ a.e.

Def: A probability preserving system $(\Omega, \mathcal{B}, \mu, T)$

is called ergodic if every T-inv function $f : \Omega \to \mathbb{R}$

is equal to a constant function a.e.

[i.e. if $\exists c$ s.t. $\mu[f \neq c] = 0$]

Birkhoff's Pointwise Ergodic Theorem

Supose $(\Omega, \mathcal{B}, \mu, T)$ is a p.p. system and let $f: \Omega \to \mathbb{R}$ be a bounded measurable function. Then

$$\bar{f}(\omega) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f(T^k \omega) \quad \text{exists a.e.}$$

If $(\Omega, \mathcal{B}, \mu, T)$ is ergodic, then $\bar{f}(\omega) = \int_\Omega f \, d\mu$ a.e.

Proof: First, some reductions

(1) $|f| \le 1$

(2) $\underline{0 \le f \le 1}$ $\qquad f = \underbrace{f \cdot 1_{[f\ge 0]}}_{f^+} - \underbrace{|f| 1_{[f<0]}}_{f^-} = f^+ - f^-$

We will prove that $\bar{A}(\omega) := \limsup_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f(T^k \omega)$

will equal $\underline{A}(\omega) := \liminf_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N-1} f(T^k \omega)$

We have to prove that $\bar{A}(\omega) = \underline{A}(\omega)$ a.e.

Notice: $\bar{A} \ge \underline{A}$, So $\bar{A} = \underline{A} \iff \mu[\bar{A} > \underline{A}] = 0$

$\iff \forall n \ \mu[\bar{A} - \underline{A} > \frac{1}{n}] = 0$

(This uses the fact that $[\bar{A} - \underline{A} > \frac{1}{n}] \uparrow [\bar{A} - \underline{A} > 0]$

We'll show $\int (\bar{A} - \underline{A}) \, d\mu = 0$ becau-se this will imply

$\mu[\bar{A} - \underline{A} > \frac{1}{n}] = \int 1_{[\bar{A} - \underline{A} > \frac{1}{n}]} d\mu \le \int n(\bar{A} - \underline{A}) \, d\mu$

• We start proving $\int (\hat{A} - \underline{A})\, d\mu = 0$

Fix $C > 0$ very big, $\varepsilon > 0$ very small.

Define $\tau(\omega) = \min \left\{ n > 0 \;\middle|\; \frac{1}{n} \sum_{k=0}^{N-1} f(T^k \omega) > \bar{A}(\omega) - \varepsilon \right\}$

(this is well defined by the def of limsup)

We "color" the time interval $0, 1, \dots, N$ as follows

(1) <u>$k=0$</u> If $\tau(\omega) > C$, color k "<u>red</u>"

If $\tau(\omega) \leq C$, color $0, 1, \dots, \tau(\omega) - 1$ "<u>blue</u>"

Move to the next k

(2) If $\tau(T^k \omega) > C$, color k "<u>red</u>". Otherwise

Otherwise color the next $\tau(T^k \omega)$ k's "blue".

Move to the next k.

(3) Stop when all k's are coloured, or you want to color a segment blue, but there are not enough k's left

We now estimate

$$\sum_{k=0}^{N-1} \left(f(T^k \omega) + 1_{\{\tau \geq C\}} (T^k \omega) \right)$$

[illegible]

Contribution of the red k's : If k is red , $T^k w \in [\tau > c]$

$$\text{So} \quad \sum_{k \text{ red}} \Big(\underbrace{f(T^k w)}_{\geq 0} + 1_{[\tau \geq c]}(T^k w) \Big) \geq (\# \text{reds})$$

$$\geq (\# \text{reds}) \, (\underbrace{\bar{A}(w) - \varepsilon}_{<1})$$

Contribution of blue k's

$$\sum_{\substack{\text{blue} \\ \text{segments}}} \Big(\sum_{k \in \text{segment}} \big(f(T^k w) + \underbrace{1_{[\tau > c)}(T^k w)}_{\geq 0} \big) \Big)$$

$$\geq \sum_{\substack{\text{blue} \\ \text{segments}}} \Big(\sum_{k \in \text{segment}} f(T^k w) \Big)$$

$$\geq \sum_{\substack{\text{blue} \\ \text{segments}}} \text{length} \times \big(\underbrace{\bar{A}(T^{\text{begining}} w)}_{= \bar{A}(w)} - \varepsilon \big)$$

$$\geq \Big(\sum_{\text{segments}} \text{length} \Big) \big(\bar{A}(w) - \varepsilon \big) = (\# (\text{blue})) \, (\bar{A}(w) - \varepsilon)$$

$$\text{So} \quad \sum_{k=0}^{N-1} \big(f(T^k w) + 1_{[\tau \geq c]}(T^k w) \big) \geq (\# \text{reds} + \# \text{blues}) (\bar{A}(w) - \varepsilon)$$

$$\geq (N - C)(\bar{A}(w) - \varepsilon)$$

$$\text{thus} \quad \frac{1}{N} \sum_{k=0}^{\infty} \big(f(T^k w) + 1_{[\tau > c)}(T^k w) \big) \geq \Big(1 - \frac{C}{N}\Big) (\bar{A}(w) - \varepsilon)$$

→ By the monocity and linearity of the integral comes

$$\frac{1}{N} \sum_{k=0}^{N-1} \Big(\int_\Omega f \circ T^k \, d\mu + \int_\Omega 1_{[\tau > c)} \circ T^k \, d\mu \Big) \geq \Big(1 - \frac{C}{N}\Big) \Big(\int_\Omega \bar{A} \, d\mu - \varepsilon \Big)$$

By Assumption, μ is T-invariant,

<u>Ex</u> : Show $\int g \cdot T^k = \int g$ for all g bounded measurable (assuming μ is T-invariant)

By the exercise, $\frac{1}{N} \sum_{k=0}^{N-1} \left(\int_\Omega f \, d\mu + \mu[\tau > c] \right) \geq \left(1 - \frac{C}{N}\right) \left(\int_\Omega \bar{A} \, d\mu - \varepsilon \right)$

$$\Rightarrow \int_\Omega f \, d\mu + \mu[\tau > c] \geq \left(1 - \frac{C}{N}\right) \left(\int_\Omega \bar{A} \, d\mu - \varepsilon \right)$$

Taking $N \to \infty$, $\int_\Omega f \, d\mu + \mu[\tau > c] \geq \int_\Omega \bar{A} \, d\mu - \varepsilon$

Taking $C \to \infty$ (and using $[\tau > c] \downarrow \phi$)

$$\int_\Omega f \, d\mu \geq \int_\Omega \bar{A} \, d\mu - \varepsilon$$

Taking $\varepsilon \to 0$ comes $\int_\Omega f \, d\mu \geq \int_\Omega \bar{A} \, d\mu$.

<u>Ex</u> : Redefine τ and reach:

$$\int_\Omega \underline{A} \, d\mu \geq \int_\Omega f \, d\mu$$

- So $\int_\Omega (\bar{A} - \underline{A}) \, d\mu \leq 0$

but $\bar{A} - \underline{A} \geq 0 \Rightarrow \int_\Omega \bar{A} - \underline{A} \, d\mu \geq 0$

So $\int_\Omega (\bar{A} - \underline{A}) \, d\mu = 0$ //

□

It's easy to see that $\bar{f}(\omega) = \lim \frac{1}{N} \sum_{k=0}^{N-1} f(T^k \omega)$

is T-invariant ($\bar{f} \circ T = \bar{f}$).

So in the ergodic case $\exists\, c$ s.t. $\bar{f}(\omega) = c$ a.e.

We now calculate c :

$$C = \lim \frac{1}{N} \sum_{k=0}^{N-1} f(T^k \omega)$$

so $$C = \int_\Omega c \, d\mu = \int_\Omega \lim_{n \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} f(T^k \omega) \, d\mu$$

$$= \lim_{n \to \infty} \int \frac{1}{N} \sum_{k=0}^{N-1} f(T^k \omega) \, d\mu$$

$$= \lim_{n \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \int_\Omega f \circ T^k \, d\mu$$

$$= \int_\Omega f \, d\mu .$$