BASIC ERGODIC THEORY: EXERCISES

Exercises for Lecture 1

Exercise 1.1. Suppose $(\Omega, \mathscr{F}, \mu)$ is a probability space. Show:

(1) If $A, B \in \mathscr{F}$ and $A \subset B$, then $\mu(A) \leq \mu(B)$
(2) If $A, B \in \mathscr{F}$ then $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$

Exercise 1.2. Suppose $A_i \in \mathscr{F}$ and $A_i \subset A_{i+1}$ for all i. Show that $\mu(A_n) \xrightarrow[n\to\infty]{} \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$. (Hint: Write $\bigcup_{n=1}^{\infty} A_n = \biguplus_{n=1}^{\infty} B_n$ for suitable disjoint sets B_n.)

Exercise 1.3. Suppose $A_i \in \mathscr{F}$ and $A_i \supset A_{i+1}$ for all i. Show that $\mu(A_n) \xrightarrow[n\to\infty]{} \mu\left(\bigcap_{n=1}^{\infty} A_n\right)$.

Exercise 1.4. A subset $U \subset \mathbb{R}$ is called *open*, if for every $x \in U$ there is a $\delta > 0$ s.t. for every y, $|y - x| < \delta \Rightarrow y \in U$. Show:

(1) Every open set is union of open intervals
(2) Every open set is a *countable* union of intervals
(3) If a σ-algebra $\mathscr{F}$ contains all open intervals, then it contains all open sets.
(4) If a σ-algebra $\mathscr{F}$ on $[0, 1]$ contains all open intervals, then every continuous function $f : [0, 1] \to \mathbb{R}$ is $\mathscr{F}$-measurable

Exercises for Lecture 2

Exercise 2.1. A *simple function* is a finite linear combination of indicators of measurable sets. Show that every simple function is measurable.

Exercise 2.2. Suppose f_n are measurable functions for all $n \in \mathbb{N}$, and there is a constant M s.t. $-M \leq f_n \leq M$ for all n. Show that $f(\omega) := \sup_{n\in\mathbb{N}} f_n(\omega)$, $g(\omega) := \inf_{n\in\mathbb{N}} f_n(\omega)$ are measurable.

Exercise 2.3. Prove that for every bounded measurable function $|\int f d\mu| \leq \int |f| d\mu$

Exercise 2.4. We say that two bounded measurable functions f, g are *equal almost everywhere* if $\mu[f \neq g] = 0$. Verify that $[f \neq g]$ is measurable. Then, show that if $f = g$ a.e., then $\int f d\mu = \int g d\mu$.

Exercise 2.5. Suppose $(X, \mathscr{B}, \mu, T)$ is a probability preserving transformation. Show that $\int f \circ T d\mu = \int f d\mu$ for all $f : X \to \mathbb{R}$ bounded measurable.

Exercise 2.6. Prove Chebyshev's Inequality: If $f : X \to \mathbb{R}$ is bounded and measurable, then $\mu[f > t] \leq \frac{1}{t^2} \int f^2 d\mu$ for all $t > 0$.

1. Exercises for Lecture 3

Exercise 3.1. Show that if f, g are bounded measurable functions satisfying $f \geq g$, then $\int (f-g)d\mu = 0$ implies that $\mu[f \neq g] = 0$.

Exercise 3.2. Show that $\overline{f}(\omega) = \limsup_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(T^k\omega)$ is T-invariant. Conclude, using the Bounded Convergence Theorem, that if $(\Omega, \mathscr{F}, \mu, T)$ is ergodic, then $\overline{f}(\omega) = \int f d\mu$ a.e.

Exercise 3.3. Show that $H(\vec{q}_1, \ldots, \vec{q}_N; \vec{p}_1, \ldots, \vec{p}_N)$ is an invariant function for the gas system discussed in lecture 1.

Exercise 3.4. Suppose μ_1, μ_2 are two *different* ergodic invariant probability measures for the same dynamical system $T : \Omega \to \Omega$. Prove that there are disjoint measurable sets A, B s.t. $\mu_1(A) = 1$, $\mu_2(B) = 1$, $\mu_1(B) = 0$, $\mu_2(A) = 0$.

Exercise 3.5. Complete the proof of the ergodic theorem by showing that $\int f d\mu \leq \int \underline{A} d\mu$, where $\underline{A}(\omega) = \liminf\limits_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(T^k\omega)$.

2. Exercises for Lecture 4

Exercise 4.1. Show that the angle doubling map $T(x) = 2x \bmod 1$ is probability preserving on $[0,1]$ with the Borel sets $\mathscr{F}$ and the Lebesgue measure μ. Guidance: show that $\mathscr{A} := \{E \in \mathscr{B} : T^{-1}E \in \mathscr{B} \text{ and } \mu(T^{-1}(E)) = \mu(E)\}$ is a σ-algebra which contains the intervals.

Exercise 4.2. Prove the approximation lemma: Suppose E is a Borel subset of $[0,1]$, then for every $\varepsilon > 0$ there is a finite collection of dyadic intervals $I_1, \ldots, I_N$ s.t. $\mu(E \triangle \bigcup_{k=1}^{N} I_k) < \varepsilon$, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$. Guidance: Show that

1. The statement is true for $E = [a, b)$
2. The statement is true for $E = \{a\}$
3. The statement is true for every interval
4. If the statement is true for E, then it is true for E^c
5. If the statement is true for $E_1, E_2, \ldots, E_N \in \mathscr{F}$, then it is true for $\biguplus_{i=1}^{N} E_i$.
6. If the statement is true for $E_1, E_2, \ldots, \in \mathscr{F}$, then it is true for $\biguplus_{i=1}^{\infty} E_i$.

Now deduce that $\mathscr{A} := \{E \subset [0,1) : \forall\varepsilon\exists n, I_1, \ldots, I_n \text{ dyadic s.t. } \mu(E \triangle \bigcup_{k=1}^{n} I_k) < \varepsilon\}$ is a σ-algebra which contains $\mathscr{F}$.

Exercise 4.3. Prove that $T(x) = 3x \bmod 1$ on $[0,1]$ equipped with Lebesgue's measure is mixing.

Exercise 4.4. Prove that $T(x,y) = (x + \alpha \bmod 1, y + \beta \bmod 1)$ on $[0,1)^2$ is not mixing.