X. Cabré - 4

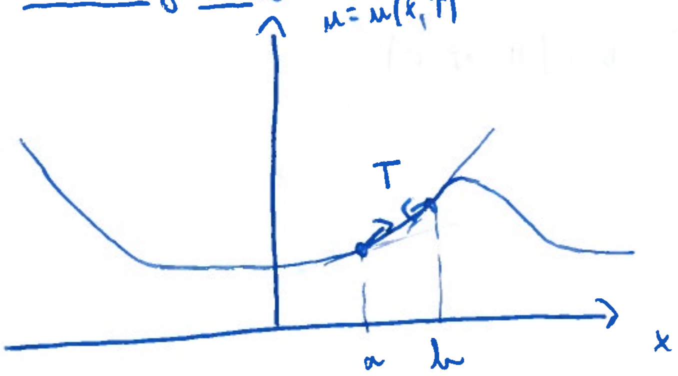
Remark: $\begin{cases} \Delta u = 0 \\ u = g \quad \text{in } \partial\Omega \end{cases}$

Maximum principle $\Rightarrow$ uniqueness

It was required that $g \in C(\Omega)$, which doesn't happen in the game $g = \begin{cases} 1 \text{ in } \Gamma_0 \\ 0 \text{ in } \Gamma_c \end{cases}$.

Uniqueness still happens, but not as easily.

Vibrating string



$x \in \mathbb{R}$

$t \in \mathbb{R}$ Time

Modelling:

· Newton law $F = m \cdot a$. Assume homogeneity of string: mass density $= 1 g\, m^{-1}$

$$a = u_{tt}$$

We have a tension force between a and b given by

$$u_x(b,t) - u_x(a,t)$$

Take $b = a+h$, $h \to 0$.

$$\underbrace{\int_a^{a+h} u_{tt}\, dx}_{\text{Newton force}} = u_x(x,t)\Big|_{x=a}^{x=a+h} = \int_a^{a+h} u_{xx}\, dx$$

$$\Downarrow$$

$$\frac{1}{h}\int_a^{a+h} (u_{tt} - u_{xx})\, dx = 0$$

$$\Downarrow$$

$u_{tt} - u_{xx} = 0 \quad \longleftarrow$ Wave equation in 1D.

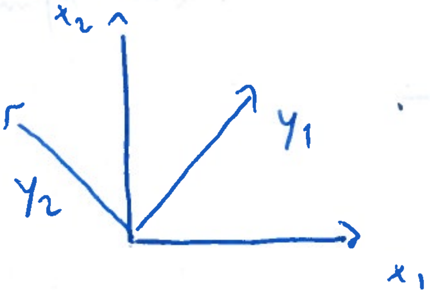
Second order PDE.

Wave equation: $u_{x_1x_1} - u_{x_2x_2} = 0$

Laplace equation: $u_{x_1y_1} + u_{x_2x_2} = 0$

Their solutions behave ~~completely~~ very differently.

write $\begin{cases} y_1 = \dfrac{x_1 + x_2}{\sqrt{2}} \\[2ex] y_2 = \dfrac{x_1 - x_2}{\sqrt{2}} \end{cases}$



The wave equation then writes as

$$u_{y_1y_2} = 0 \Rightarrow u_{y_1} = \phi(y_1) \Rightarrow \int u_{y_1}(y_1, y_2)\, dy_1 = \int \phi(y_1)\, dy_1$$

$$\Rightarrow u(y_1, y_2) = \tilde{\phi}(y_1) + ct(y_2)$$

$\tilde{\phi} = \int \phi$ integration constant for each y_2.

General solution: $u(y_1, y_2) = \tilde{\varphi}(y_1) + \tilde{\psi}(y_2)$

$$= \varphi(x_1 + x_2) + \psi(x_1 - x_2)$$

__Cauchy Problem__

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(x,0) = g(x) & \leftarrow \text{Initial profile} \\ u_t(x,0) = h(x) & \leftarrow \text{Initial impulse} \end{cases}$$

__Ex__: Compute φ and ψ in terms of g and h.

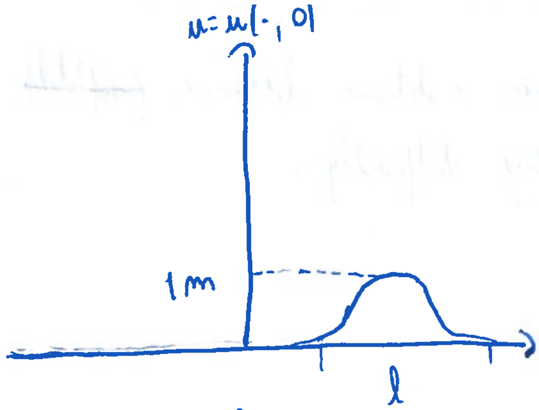
__Case__: $h \equiv 0$

$$u(x,t) = \frac{1}{2}\left(g(x+t) + g(x-t)\right)$$

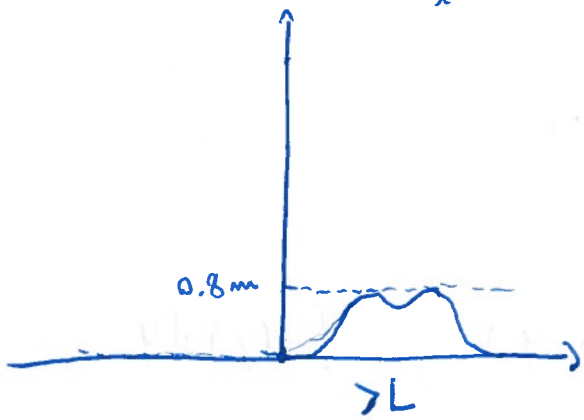
$u(x,0) = g(x)$

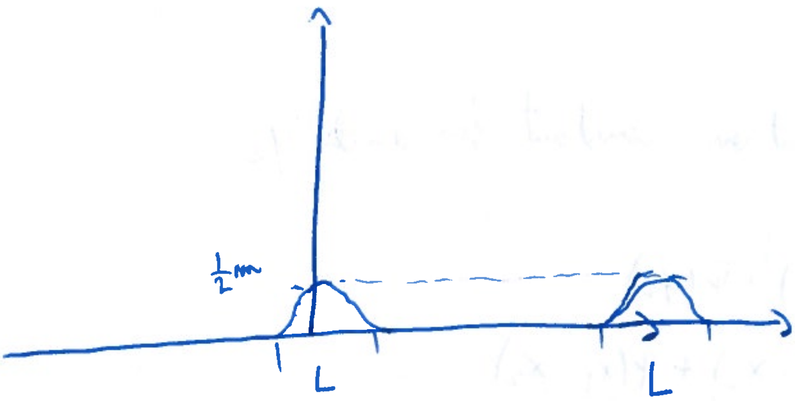
$u_t(x,0) = \frac{1}{2}\left(g'(x) - g'(x)\right) = 0 = h(x)$.

$t=0$



The wave splits in 2 moving with symmetric velocities.





Regularity of PDEs

Sols of wave equation are C^2 but not more in general.

" Laplace " are always C^∞.

d'Alembert trick

$$\begin{cases} x = x \\ t = iy \end{cases} \qquad u_{tt} - u_{xx} = 0 \iff \cancel{u_{xx} + u_{yy} = 0} \quad \overset{\Delta u}{\overset{\shortparallel}{u_{yy} + u_{xx}}} = 0$$

Laplace equation.

General solution $\Delta u = 0$ should be

$$u(x,y) = \varphi(x+iy) + \psi(x-iy) = \varphi(z) + \psi(\bar{z})$$

Consider $u(x,y) = \varphi(z)$. When is φ differentiable as a complex function?

Cauchy-Riemann equations

$\varphi \overset{\mathbb{C}\to\mathbb{C}}{\ }$ differentiable $\Rightarrow \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

$\varphi(z) = u(z) + i\,v(z)$

with Cauchy Riemann $\quad u_{xx} = v_{yx} = v_{xy} = -u_{yy} \Rightarrow \Delta u = 0.$

If φ dif. in $\Omega \subseteq \mathbb{C}$ then φ is analytic.

- The real part & the imaginary part of analytic complex function are harmonic $\mathbb{R}^2$ functions.

In $\mathbb{C}$ an analytic function is smooth $(C^\infty) \longrightarrow$ every harmonic function $(\mathbb{R}^2)$ is C^∞!

Ex: $\varphi(z) = (x+iy)^3 = x^3 - 3xy^2 + i(\dots)$

$\Downarrow$

$u(x,y) = x^3 - 3xy^2$ is a Harmonic function.

$\varphi(z) = e^z = e^{x+iy} = e^x(\cos y + i \sin y)$

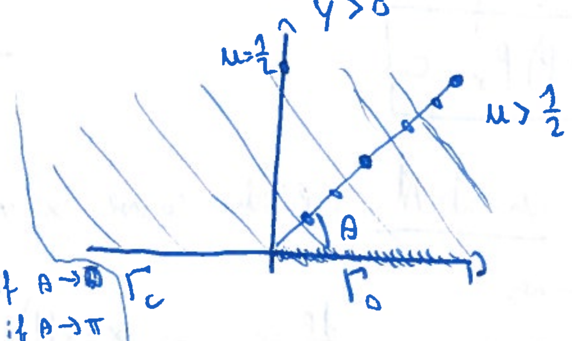
$\Downarrow$

$u(x,y) = e^x \cos y$ is Harmonic

$\varphi(z) = \operatorname{Log} z = \operatorname{Log}(r e^{i\theta}) = \operatorname{Log} r + i\theta = \operatorname{Log}|z| + i \arg z.$

$\Downarrow$

$u = \log r = \log\sqrt{x_1^2 + x_2^2}$ and $v = \theta = \arctan\frac{x_1}{x_2}$ are Harmonic.



Finance game in $\Omega = \mathbb{R}^2_+$

$u(0,y) = \frac{1}{2}$

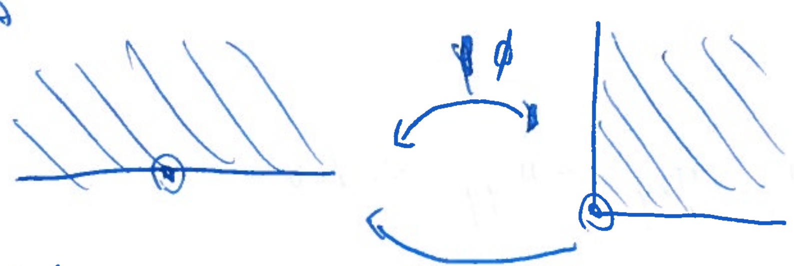
u must depend only of θ

$u = \frac{1}{2}\left(1 + \frac{2}{\pi}\arctan\frac{x}{y}\right)$ $\quad$ (we $u = \begin{cases} 1 & \text{if } \theta \to 0 \\ 0 & \text{if } \theta \to \pi \end{cases}$ and $u(0,y) = \frac{1}{2}$)

Conformal transformation

Transformations that change the domain

e.g.



In this case $\phi : z \to z^2$.

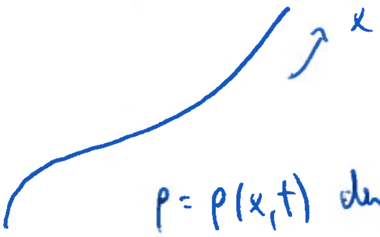
u Harmonic

$\Downarrow$

φ analytic

We can get solutions on different domains

Talk-2



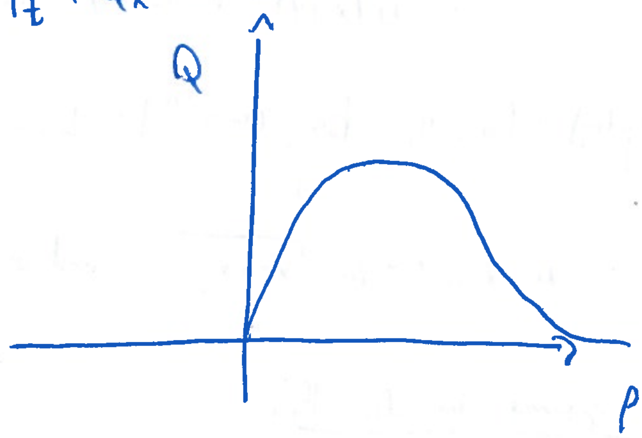
$\rho = \rho(x,t)$ density $\quad u(x,t)$ velocity $\qquad Q = \rho u$.

Conservation of cars:

$$\int_{x_1}^{x_2} \rho(x,t_2)\,dx - \int_{x_1}^{x_2} \rho(x,t_1)\,dx = \int_{t_1}^{t_2} \left[Q(x_1,t) - Q(x_2,t) \right] dt$$

$$\Downarrow$$

$$\rho_t + Q_x = 0$$



We assume that $Q(x,t) = Q(\rho)$

Equation transforms in

$$\boxed{\rho_t + Q'(\rho)\,\rho_x = 0}$$

Method of characteristics: consider curve $x = x(t)$.

Eq. becomes

$\frac{dx}{dt} = Q'(\rho)$, $\quad \frac{d\rho}{dt} = 0 \implies x = x(t)$ is a line.