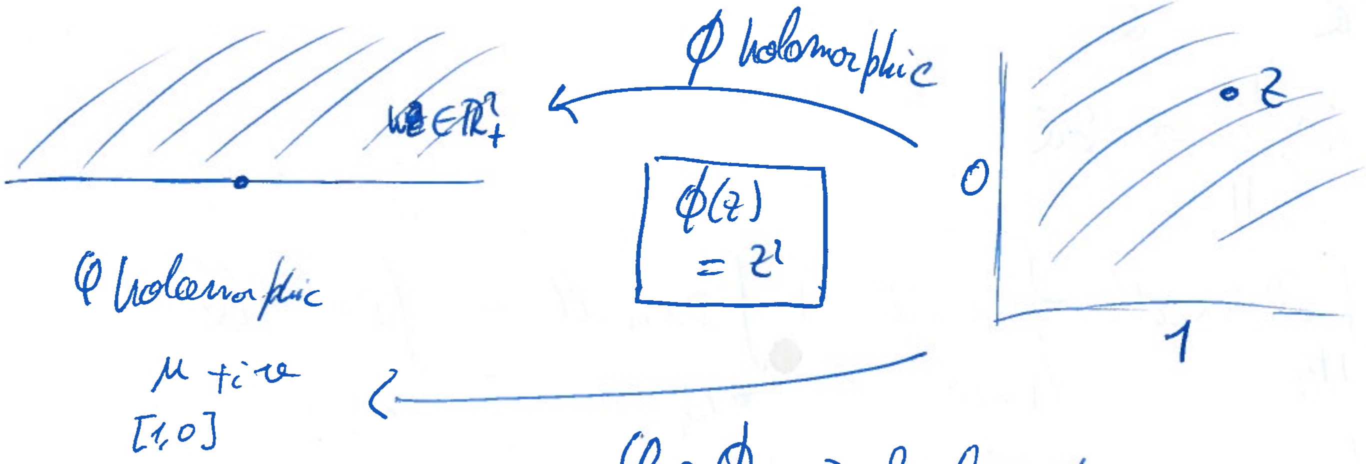
Conformal Transformations



$\varphi \circ \phi \longrightarrow$ Real part

$z = re^{i\theta}$, $0 < \theta < \pi/2$

$z^2 = r^2 e^{i(2\theta)}$, $0 < 2\theta < \pi$

The heat equation Fourier series / separation of variables

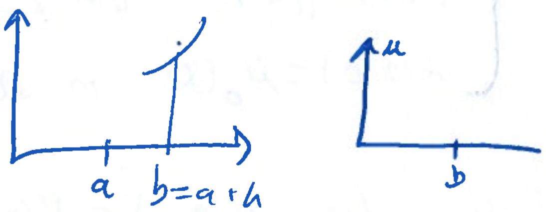
[L-Cable]

Fourier law of heat

$n = 1$

$x \in \mathbb{R}$
$t \in \mathbb{R}$ time $\Big\}\ u(x,t)$

the heat flux is proportional to the temperature gradient.



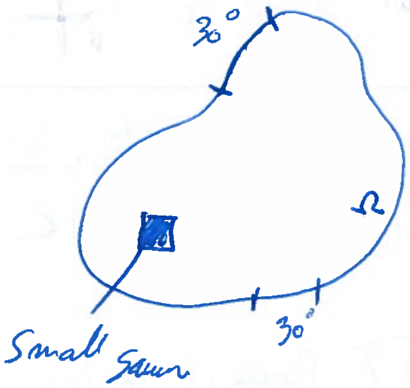
Flux of heat at $x = b = D\, u_x(b,t)$

$D = \text{cte} > 0$

Total heat of the interval (a,b) at time t $\qquad n \geq 2$

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = D\left[u_x(b,t) - u_x(a,t)\right]$$

$$\int_a^b u_t(x,t)\,dx = \int_a^b D\,u_{xx}(x,t)\,dx$$



$$\frac{d}{dt}\int_Q u\,dx = \int_Q u_t\,dx$$

$\parallel$

flux across ∂Q

$\parallel$

$$\underbrace{\int_{\text{Right}} D u_x\,dl - \int_{\text{left}} D u_x\,dl}_{D\int_Q u_{xx}} + \underbrace{\int_{\text{Top}} D u_y\,dl - \int_{\text{Bottom}} D u_y\,dl}_{D\int_Q u_{yy}}$$

$\Downarrow$

$$u_t = D\Delta u = D\Delta_{xy} \leftarrow \text{Spatial Laplacian}$$

<u>Initial boundary value problem</u>

$$\text{IBVP}\begin{cases} u_t = D\Delta u & \text{in } \Omega \times (0,\infty) \\ u(x,t) = g(x) & \text{on } \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases}$$

We will see $\lim_{t\to\infty} u(x,t) = V(x) = ?$ where $\begin{cases}\Delta V(x) = 0 \\ V = g \text{ on } \partial\Omega\end{cases}$

Does not depend on time.

heat = probability = expected game

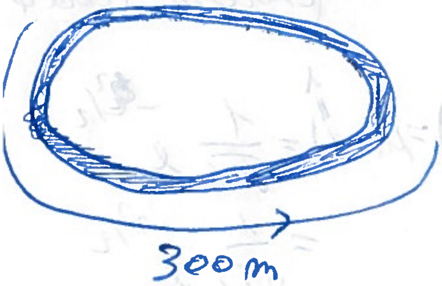
when we discretize

$V(e) = \frac{1}{4}(\cdots)$.

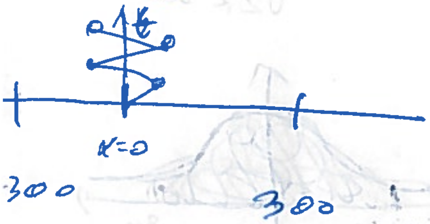
<u>Mean value property for harmonic functions, $\forall n$</u>

$\overline{B_R}(x_0) \subset \Omega \subset \mathbb{R}^n$
$$V(x_0) = \frac{1}{|\partial B_R|}\int_{\partial B_R(x_0)} V = \frac{1}{|B_R|}\int_{B_R(x_0)} V$$

$\boxed{n=2}$ Prove it using the div. Theorem.



Prob. that before 1 hour given ~~sefan~~ at least a turn and has come back to me already?



Descretize and take a time step

$\Delta x = h$

$\Delta t = \tau$

Find time = 1 hour

1 hour $= k\tau$, $k \in \mathbb{Z}$ large

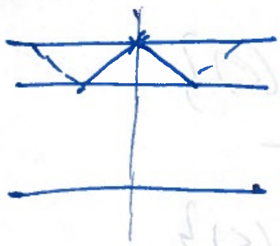
Position of the ball after k time steps $= hX_1 + hX_2 + \dots + hX_k$

$X = \begin{cases} 1 & , pr = 1/2 \\ -1 & , pr = 1/2 \end{cases}$ $\quad = h \sum_i^k X_i$

Random Vari… independ… identically distributed

Continuous prob.? $h \to 0 \quad \tau \to 0$

$$u(x, t+\tau) = \frac{1}{2}\left(u(x+h, t) + u(x-h, t)\right)$$



Subtract $u(x,t)$ and divide h^2

We have to divide by cte τ $\quad \tau = \frac{h^2}{2D}$

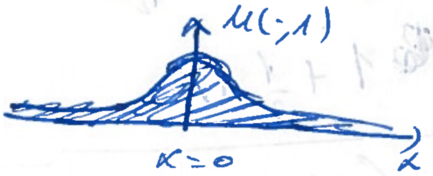
$$\boxed{u_t = D u_{xx}}$$

$t = 1 = k\tau$, $D = 1/2$

$h = \sqrt{\tau} = \frac{1}{\sqrt{k}}$ $\longrightarrow$ Position

$\longleftarrow$ Prob $= \frac{x_1 + \dots + x_k}{\sqrt{k}}$

law with density $u(x, t=1)$



$\begin{cases} u_t = \frac{1}{2} u_{xx} & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = \delta_{0=x} \end{cases}$

Change of Variables

$\bar{x} = \lambda x \qquad \xi = \frac{x}{\sqrt{t}} = \frac{\bar{x}}{\sqrt{\bar{t}}}$

$\bar{t} = \lambda^2 t$

and then: $\begin{cases} u_{\bar{t}} = \frac{1}{2} u_{\bar{x}\bar{x}} \\ u(x,0) = \delta_{0=x} \end{cases}$

Ansatz,

$u(x,t) = \varphi\left(\frac{x}{\sqrt{t}}\right) \leftarrow$ Not the correct Ansatz.

Not normalized!

$\int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} \varphi\left(\frac{x}{\sqrt{t}}\right) dx$

$= \left| \frac{x}{\sqrt{t}} = \xi \right| = \int \varphi(\xi)\,d\xi \sqrt{t}$ $\quad \swarrow$ depends on Time!

the solution should not …

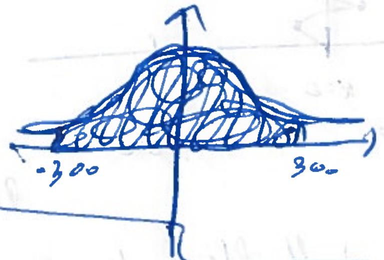
correct Ansatz : $\frac{1}{\sqrt{t}}\varphi\left(\frac{x}{\sqrt{t}}\right)$

Exercise. (cont.)

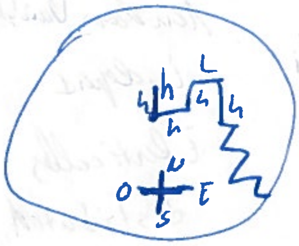
$$\varphi(\xi) = \mu(x,1) = \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Central limit theorem

$$\frac{x_1 + \dots + x_n}{\sqrt{n}} \xrightarrow{\text{probability}} N(0,1) = \text{Gaussian}$$



u = length until the game is finished



$$u(e) = h + \frac{1}{4}\left(u(E) + u(N) + u(W) + u(S)\right)$$

$$0 = -4u(e) + 4h + u(E) + u(W) + u(S) + u\dots$$

$$0 = \frac{4h}{h^2} + \frac{\{u(E) + u(W) - 2u(e)\}}{h^2} + \frac{\{u(N) + u(S) - 2u(S)\}}{h^2}$$

$h \to 0$

Problem

$$0 = \infty + \Delta u$$

We have to put h^2 in the numerator :

measure of 1 step

$$0 = \frac{4h^2}{2h^2} + \frac{\{\dots\}}{h^2}$$

$$0 = 1 + \frac{1}{2}\Delta u$$

$$\begin{cases} -\frac{1}{2}\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$\begin{cases} -u_{xx} = \lambda u \\ x \in (0,\pi) \\ u(0) = u(\pi) = 0 \end{cases}$$

The eigenfunctions of the Laplacian are $u(x) = \sin(nx)$, $n = 1, 2, 3, 4, \dots$

$$\sum_{n=0}^{\infty} c_n e^{-nt} \sin(nx)$$

$$\boxed{\sum_{n=0}^{\infty} c_n \sin(nx)} = u(x,0)$$