Probability game

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T1

$\mathbb{R}^2$

$\cup$

Ω open set

Particle starting always same $x \in \Omega$ moves randomly

Random:

① I have no memory

② I do not privilege any direction

Brownian motion

$\partial\Omega = \Gamma_c \cup \Gamma_0 \qquad \Gamma_c \cap \Gamma_0 = \emptyset$

Γ_c = closed part

Γ_0 = open part

First time the trajectory hits the boundary

$\left< \begin{array}{l} \text{in } \Gamma_c \\ \text{or} \\ \text{in } \Gamma_0 \end{array} \right.$

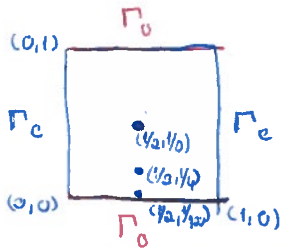
$u(x)$ = probability of exiting the domain when I start at the point $x \in \Omega$ = ? $\in [0,1]$

$u : \Omega \to [0,1]$

Example 1

$\Omega = (0,1) \times (0,1)$

$x = (1/2, 1/2)$



$u(1/2, 1/2) = 1/2$

$u(1/2, 1/4) = ?$

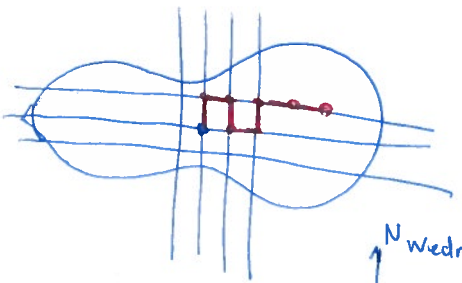
$u(1/2, 1/100) > 1/2$

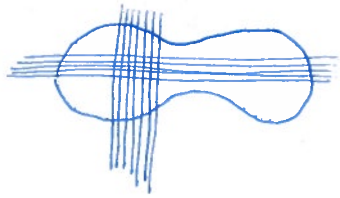
How to "solve" the problem?

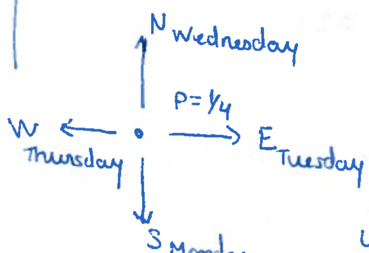
We discretize the problem

We pick a step $h > 0$ very small

At the end $\boxed{h \to 0}$







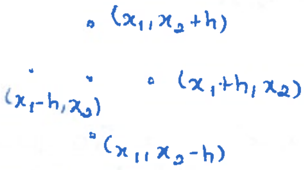
$\boxed{u^h}$

$u^h(x) = u^h(E) \cdot \text{prob}(x \to E) + u^h(W) \cdot \text{prob}(x \to W) + u^h(N) \cdot \text{prob}(x \to N) + u^h(S) \cdot \text{prob}(x \to S)$

conditional prob.

$u^h(x) = 1/4 \left(u^h(E) + u^h(W) + u^h(N) + u^h(S) \right)$

$0 = u^h(E) + u^h(W) - 2u^h(x) + u^h(N) + u^h(S) - 2u^h(x)$



$0 = u^h(x_1+h, x_2) + u^h(x_1-h, x_2) - 2u^h(x) + u^h(x_1, x_2+h) + u^h(x_1, x_2-h) - 2u^h(x)$

$u^h = v$

$v(x_1+h) + v(x_1-h) - 2v(x_1)$

$= \{v(x_1+h) - v(x_1)\} - \{v(x_1) - v(x_1-h)\}$

$(h \to 0) \simeq h\, v'(x_1) - h\, v'(x_1-h)$

$= h \{v'(x_1) - v'(x_1-h)\} \simeq h(h(v'')(x_1)) = h^2 v''(x_1)$ as $h \downarrow 0$

$0 = u_{x_1x_1}(x) + u_{x_2x_2}(x)$

where: $u_{x_1x_1} = \frac{\partial^2 u}{\partial x_1^2} = \partial_{x_1x_1} u = \partial^2_{x_1} u$, u_{11} (notation)

$u_{x_1x_2} = (u_{x_1})_{x_2} = \lim_{h\to 0} \frac{u_{x_1}(N) - u_{x_1}(X)}{h} = \lim_{h\to 0} \frac{u_{x_1}(x_1, x_2+h) - u_{x_1}(x_1,x_2)}{h}$

$= \lim_{h\to 0} \frac{\frac{u(x_1+h, x_2+h) - u(x_1, x_2+h)}{h} - \frac{u(x_1+h, x_2) - u(x_1,x_2)}{h}}{h}$

$= \lim_{h\to 0} \frac{1}{h^2}\{u(x_1+h, x_2+h) - u(x_1+h, x_2) - u(x_1, x_2+h) + u(x_1,x_2)\}$

$u_{x_1x_2} = u_{x_2x_1}$

For every $x \in \Omega$, I should have

$u_{x_1x_1}(x) + u_{x_2x_2}(x) = 0$

$\Delta u(x) \to$ the laplacian of u at the point x

$u = u(x_1, x_2, \dots, x_n)$

$\Delta u = u_{x_1x_1} + \dots + u_{x_nx_n}$ <u>Laplacian</u>

<u>Examples</u>

- $u(x) = x_1^2 + x_3^3$

  $u_{x_1} = 2x_1$, $u_{x_1x_1} = 2$, $u_{x_3} = 3x_3^2$, $u_{x_3x_3} = 6x_3$ $\Big\}$ $\Delta u = 2 + 6x_3$

- $u(x) = u(x_1, x_2) = x_1^2 - x_2^2$ $\qquad \Delta u = 0$

<u>Definition</u>: u is harmonic in Ω iff $u \in C^2(\Omega)$ and $\Delta u = 0 \;\forall x \in \Omega$

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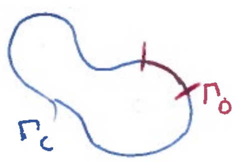
T2

06/07/2016

- Laplacian. Maximum principle.
- Wave equation. d'Alembert.
- Complex variables (conformal transformations)
- Heat equation & Fourier series
- (- Hilbert space methods)



$u(x)$ - expected gain starting from x
= probability of existing

$\Delta u = 0$ in Ω

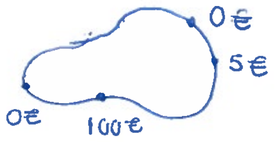
u is a harmonic function

$\Delta u = u_{x_1x_1} + u_{x_2x_2} + \dots + u_{x_nx_n}$

- Boundary conditions

$$\begin{cases} \Delta u = 0 \\ u = \begin{cases} 1 & \text{on } \Gamma_0 \\ 0 & \text{on } \Gamma_c \end{cases} \end{cases}$$ Boundary value problem

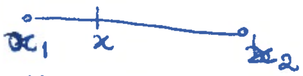
$u : \bar{\Omega} \to \mathbb{R}$



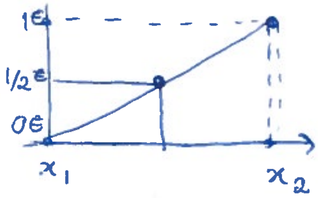
Given $g : \partial\Omega \to \mathbb{R}$

(BVP) $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$ Dirichlet problem

$\boxed{n=1}$



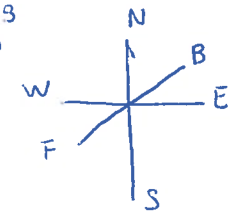
$u_{xx} = 0$
$u_x = b$ (const.)
$u = ax + b$



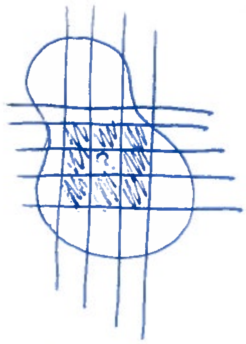
$u(x) = \frac{x - x_1}{x_2 - x_1}$

$\mathbb{R}^2$ same argument
in $\mathbb{R}^n$ same
$\downarrow$
Δ in $\mathbb{R}^n$

$\mathbb{R}^3$



$x_{11} + x_{22} + x_{33} = 0$



$P_{ij} = (ih, jh)$ $i, j \in \mathbb{Z}$

Defn P_{ij} interior $\iff$ 4 neighbors $\in \Omega$
P_{ij} boundary pt. otherwise

$u^h_P = u^h(P_{ij})$

$$\Delta_h u^h(P_{ij}) := 1/h^2 \{u(P_{i+1,j}) + u(P_{i-1,j}) + u(P_{i,j+1}) + u(P_{i,j-1}) - 4(P_{ij})\}$$

$$= 4/h^2 \{\text{average of } u \text{ at the 4 neighb. of } P_{ij} - u(P_{ij})\}$$

$(=0)$

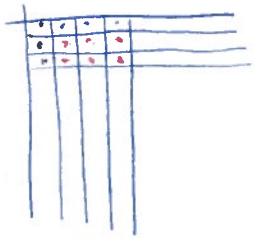
(DBVP)
$\begin{cases} \Delta^h u^h = 0 & \forall P_{ij} \text{ interior} \\ u^h = g & \forall P_{ij} \text{ boundary} \end{cases}$

Defn: u^h is an ideal image iff $\Delta_h u^h = 0$ $\forall$ interior pixel

Thm 1: $\forall\ \Omega$ b... open set, $\forall h>0$

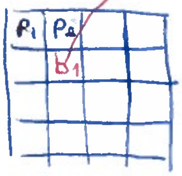
$\exists!\ u^k$ satisfying (DBVP)

Particular case



$$\begin{cases} \Delta_h u^h = 0 \\ u(\cdot) = g(\cdot) \end{cases}$$

4x4



$u(\cdot) = b_1$
unknown

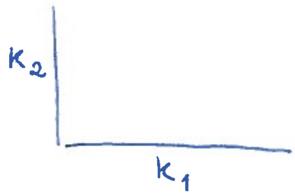
$A\vec{b} = \vec{c}$

$$A = \begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix}$$

symmetric
positive definite

$\hookrightarrow \det A \neq 0$

$k_1 \times k_2$ image $k_1, k_2 \in \mathbb{Z}^+$



$\simeq k_1 \times k_2$ int. pixels
$\hookrightarrow$ eqns

A $(k_1 \times k_2) \times (k_1 \times k_2)$ matrix

$$\begin{bmatrix} 4 & & & & & & & \\ & 4 & & & & & & \\ & & 4 & & & & & \\ -1 & 0 \cdots & -1 & 4 & -1 & 0 \cdots & -1 & \\ & & & & & & & 4 \end{bmatrix}$$

A symmetric (square matrix)
(semi-) diagonal dominant

<u>Iterative methods</u>:

Jacobi or Gauss - Seidel

$A = A_1 + A_2$

$A_1 = 4\,Id$

$$A_2 = \begin{pmatrix} 0 & & -1 \\ & \ddots & \\ & & 0 \end{pmatrix}$$

$(A_1 + A_2)\vec{b} = \vec{c}$

$4\vec{b} = \vec{c} - A_2\vec{b}$

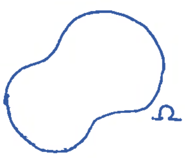
$$\vec{b}_{k+1} = \frac{\vec{c} - A_2\vec{b}_k}{4}$$

converges $\forall \vec{b}_2 (\overset{ex:}{=} 0)$

<u>Thm 1</u>

Given $\forall\ \Omega$ bound. open set in $\mathbb{R}^2$ $\forall h$, $\forall$ values of the boundary pixels.
$\exists!$ ideal image having them as boundary pixels

<u>Proof</u>:



It is enough (by linear algebra) to prove uniqueness

$\Delta_h u^h = 0$

$\Delta_h v^h = 0$

$u^h = v^h$ on boundary pts

$\Downarrow$?

$u^h = v^h$ for all int. pixels

$w^h := u^h - v^h$

$\Delta_h w^h = 0$ & $w^h = 0$ at every boundary pixel

Need to prove $w^h \equiv 0$ at interior points

Suppose not: $w^h \not\equiv 0$

or
- The largest pixel has positive value and thus it is achieved at a interior level
- The smallest value of u^h is negative and achieved at an interior point $\square$

Maximum principle

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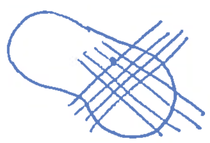
T3
05/07/2016



The Laplace operator is invariant under rotations

$\Delta u = u_{11} + \cdots + u_{nn}$
$= \underline{\text{trace}}\ D^2 u$

Hessian matrix
$= \underset{n\times n}{D^2} u = [u_{ij}] = [\partial x_i y_j u]$

Rotation in $\mathbb{R}^n$
$y = Ox$ orthonormal change of basis in $\mathbb{R}^n$
O $n\times n$ orthogonal matrix
$u(x) = \tilde{u}(y) = u(y)$
$D^2_y u = O^t D^2_x u\, O^{(t)}$ ← exercise
$\Delta_y u = \text{trace}(O^t D^2_x u\, O) = \text{trace}(O^{-1} D_x^2 u\, O)$
$= \text{trace}\ D^2_x u = \Delta_x u$

Pbs 3, 4

$r = |x| = \sqrt{x_1^2 + x_2^2}$
$= \sqrt{x_1^2 + \cdots + x_n^2}$

$u(x) = u(x_1, \ldots, x_n)$
$= u(r)$

$n = 2$ check: $u_{x_1x_1} + u_{x_2x_2} = u_{y_1y_1} + u_{y_2y_2}$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

exercise

u has radial symmetry
Laplace operator acting on radially symmetric funcs.

$\partial x_i r = \frac{\cancel{2} x_i}{\cancel{2}\sqrt{\cdots}} = \frac{x_i}{r}$

$u_{x_i} = u_r \cdot r_{x_1} = u_r \frac{x_i}{r}$

↓ exerc.

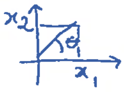
$u_{x_1x_1} = \square + \square + \square + \ldots$
$=$

$\Delta u = u_{rr} + \frac{n-1}{r} u_r$ in $\mathbb{R}^n$
& u is a func. of one real variable $r \geq 0$
PDE $\to$ ODE

$\boxed{\Delta u = r^{1-n}(r^{n-1}u_r)_r} \to$ pbs 3 & 4

$\mathbb{R}^2$
$u = u(x_1, x_2) = u(r, \theta)$ $\to$ Polar coordinates
not radially symmetric

$\begin{cases} x_1 = r\cos\theta \\ x_2 = r\sin\theta \end{cases}$



<u>Exercise</u>

$\Delta u = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2}$

$\mathbb{R}^3$ sph. coord.
$\hookrightarrow$ Wikipedia

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BVP $\begin{cases} \Delta u = 0 & \text{in } \Omega \subset \mathbb{R}^n, \ n \geq 1 \\ u = g & \text{on } \partial\Omega \end{cases}$

$g : \partial\Omega \to \mathbb{R}$ given

<u>Thm 2</u>

If Ω is an open bound. set and g is a continuous function on $\partial\Omega$, then there is at most one solution continuous in $\bar{\Omega}$ and twice diff in Ω.
(Uniqueness result) Info: $\exists$ solution
(not a simple result)

<u>Thm 3</u> (Maximum principle)

Suppose Ω open bound. in $\mathbb{R}^n$, $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and $\Delta u \geq 0$ in (i.e. u is <u>subharmonic</u>.
Then $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$

<u>Proof</u>: Take $\varepsilon > 0$
Consider
$u_\varepsilon(x) := u(x) + \varepsilon |x|^2$
$x \in \bar{\Omega} \subset \mathbb{R}^n$

<u>Claim</u>: $\max_{\bar{\Omega}} u_\varepsilon = \max_{\partial\Omega} u_\varepsilon$

<u>Proof of Claim</u>:

$\begin{cases} \partial_{x_1} u_\varepsilon(x_0) = 0 \\ \cdots \\ \partial_{x_n} u_\varepsilon(x_0) = 0 \end{cases}$

$\begin{cases} \partial_{x_1 x_1} u_\varepsilon(x_0) \leq 0 \\ \vdots \\ \partial_{x_n x_n} u_\varepsilon(x_0) \leq 0 \end{cases} \Rightarrow \Delta u_\varepsilon(x_0) \leq 0$

$\qquad \| $
$\Delta u(x_0) + 2\varepsilon n$
$2\varepsilon n$

Thus $\forall x \in \Omega$
$u(x) \leq \max_{\partial\Omega} u_\varepsilon$
$= \max_{y \in \partial\Omega} (\cancel{y+y}) u(y) + \varepsilon |y|^2)$
$\downarrow \varepsilon \downarrow 0$
$\max_{\partial\Omega} u$

Max. principle for subharm. func.

$\Downarrow$

Min. principle for subharm. fun.

$\Downarrow$

$\Delta u = 0$ in Ω

then

$$\max_{\bar{\Omega}} u = \max_{\partial u} u$$

&

$$\min_{\bar{\Omega}} u = \min_{\partial \Omega} u$$

Consequence of max. principle is the uniqueness of the BVP, even for the more general PDE $\boxed{-\Delta u = f}$ where $f : \Omega \to \mathbb{R}$ is given (non homog. eqn. eqn. with a r.h.s.)

Proof:

$$\begin{cases} -\Delta u = f \\ u|_{\partial\Omega} = g \end{cases} \qquad \begin{cases} -\Delta v = f \\ v|_{\partial\Omega} = g \end{cases}$$

$w := u - v$

$\Delta w = 0$ in Ω

w is harmonic

&

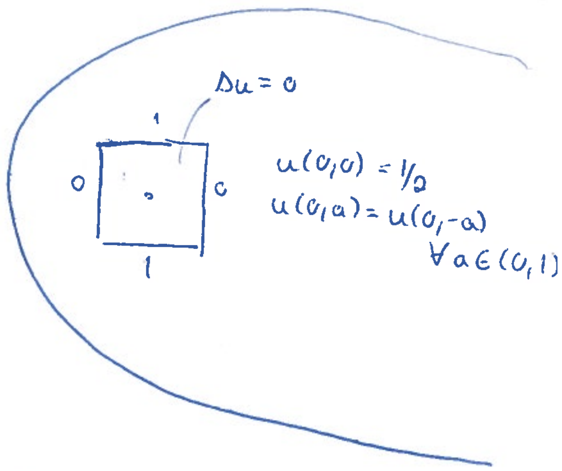
$w|_{\partial\Omega} = 0 \Rightarrow \max_{\partial\Omega} w = 0 \overset{\text{Max. princ.}}{\Longrightarrow} w = 0$ ∎

$\min_{\partial\Omega} w = 0$

Corol. 4 The sol. of Pb 3.

$$\begin{cases} \Delta u = 0 \\ u = \begin{cases} 1 & \text{if } |x| = 1 \\ 0 & \text{if } |x| = 2 \end{cases} \end{cases}$$

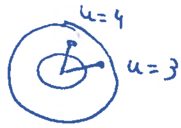
is radially symmetric; $u = u(r)$



Pf: (MP) $\Rightarrow$ uniqueness

$\Downarrow$ rotat. invariance of Δ & of the B.C.

~~sol~~ symmetry properties of B.V.P



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u satisfies $u(x_1, x_2) = u(x_1, -x_2)$

$\forall (x_1, x_2) \in (-1,1) \times (-1,1)$

i.e. u is an even function of x_2

Pf: $v(x_1, x_2) := u(x_1, -x_2)$

$$\left.\begin{aligned} v_{x_1 x_1} &= u_{x_1 x_1} \\ v_{x_2} &= (-u_{x_2}(x_1, -x_2)) \\ v_{x_2 x_2} &= u_{x_2 x_2}(x_1, -x_2) \end{aligned}\right\} \begin{aligned} &\Delta v = 0 \\ &v|_{\partial Q} = u|_{\partial Q} \end{aligned} \Rightarrow u \equiv v \ ∎$$

The Laplacian is invariant under reflections