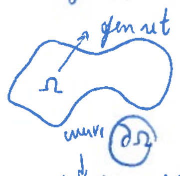
Probability game

4/7

$\mathbb{R}^2$

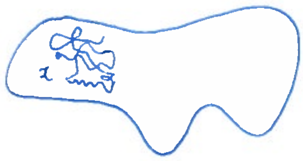
Ω



→ simple - doesn't intersect

→ close

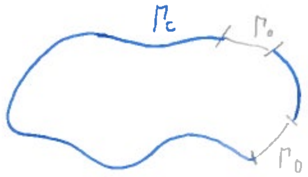
boundary of the region Ω



particle starting always from x moves randomly

1. I have no memory
2. I do not privilege any direction

Brownian motion → trajectory



$$\partial\Omega = \Gamma_C \cup \Gamma_0 \quad , \quad \Gamma_C \overset{\text{disjoint}}{\cap} \Gamma_0$$

Γ_C = closed part

Γ_0 = opened part

The game

First time the trajectory hits the boundary

- in Γ_C
- or
- in Γ_0

I want to know the probability of exiting the domain when I start at the point $x \in \Omega$

$= ? \in [0,1]$

$u(x)$, $u: \Omega \to [0,1]$

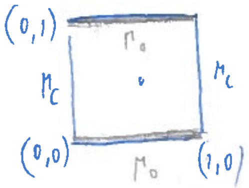
loose → 0

Win → 1

many times and the average → probability

Example 1

$\Omega = (0,1) \times (0,1)$



$x = (1/2, 1/2)$

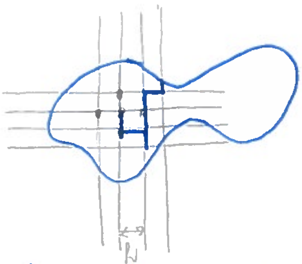
$u(1/2, 1/2) = 1/2$

$x' = (1/2, 1/100)$

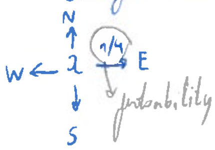
$u(1/2, 1/100) \geq 1/2$ (close to the open part ⇒ win)

How to solve the pb?

We discretize the pb $\Rightarrow$ we pick a ~~step~~ $h > 0$ very small; at the end, $h \to 0$



(4) neighbours - N, S, E, W



$$u^h(x) = u^h(E)\cdot \text{prob}(x \to E) + u^h(W)\cdot \text{prob}(x \to W) + u^h(N)\cdot \text{prob}(x \to N) + u^h(S)\cdot \text{prob}(x \to S)$$

condition probabilities

$\boxed{u^h}$ probability with this lattice (discretized)

$$u^h(x) = \frac{1}{4}\left\{ u^h(E) + u^h(W) + u^h(N) + u^h(S) \right\} \Leftrightarrow 0 = u^h(E) + u^h(W) - 2u^h(x) + u^h(N) + u^h(S) - 2u^h(x)$$

$$\Rightarrow 0 = \boxed{u^h(x_1+h, x_2) + u^h(x_1-h, x_2) - 2u^h(x_1,x_2)} + u^h(x_1, x_2+h) + u^h(x_1, x_2-h) - 2u^h(x_1,x_2) \quad (*)$$

$x_1 - h \quad x_1 \quad x_1 + h \quad x_1$

$$u^h = v, \quad v(x_1+h) + v(x_1-h) - 2v(x_1) = \{v(x_1+h) - v(x_1)\} - \{v(x_1) - v(x_1-h)\}$$

$$\underset{h\to 0}{\simeq} v'(x_1)\cdot h - v'(x_1-h)\cdot h \simeq h\,(v'(x_1) - v'(x_1-h))$$

$$\simeq h\,(h\,(v')'(x_1)) \simeq h^2 v''(x_1) \text{ as } h \to 0$$

$$(*) \quad 0 = \frac{u^h(x_1+h,x_2) + u^h(x_1-h,x_2) - 2u^h(x_1,x_2)}{h^2} + \frac{u^h(x_1,x_2+h) + u^h(x_1,x_2-h) - 2u^h(x_1,x_2)}{h^2}$$

$h \to 0$, $u^h \simeq u$

$$\boxed{0 = u_{x_1x_1}(x) + u_{x_2x_2}(x)}$$

where

$$u_{x_1x_1} = \frac{\partial^2 u}{\partial x_1^2} = \partial_{x_1x_1} u \quad \text{(notation)}$$
$$= \partial^2_{x_1} u$$
$$= u_{11}$$

$$u_{x_1x_2} = (u_{x_1})_{x_2} = \lim_{h\to 0} \frac{u_{x_1}(x) - u_{x_1}(x)}{h} = \lim_{h\to 0} \frac{u_{x_1}(x_1, x_2+h) - u_{x_1}(x_1,x_2)}{h} =$$

$$= \lim_{h\to 0} \frac{\frac{u(x_1+h, x_2+h) - u(x_1, x_2+h)}{h} - \frac{u(x_1+h, x_2) - u(x_1,x_2)}{h}}{h}$$

$$\lim_{h\to 0} \frac{1}{h^2}\left\{ u(x_1+h, x_2+h) - u(x_1, x_2+h) - u(x_1+h, x_2) - u(x_1,x_2) \right\}$$

$u_{x_1x_2} = u_{x_2x_1}$

—

For every $x \in \Omega$, I should have $\boxed{u_{x_1x_1}(x) + u_{x_2x_2}(x) = 0}$

$\Delta u(x) \to$ Laplacian of u at the point x

$u = u(x_1, x_2, x_3, \dots, x_N)$, $\Delta u = u_{x_1x_1} + u_{x_2x_2} + \dots + u_{x_nx_n}$

operator $\to$ gives a value at a point

Examples

- $u(x) = x_1^2 + x_2^3$

$$\left.\begin{aligned} u_{x_1} &= 2x_1 \\ u_{x_1x_1} &= 2 \\ u_{x_2} &= 3x_2^2 \\ u_{x_2x_2} &= 6x_2 \end{aligned}\right\} \quad \Delta u = 2 + 6x_2$$

- $u(x) = u(x_1, x_2) = x_1^2 - x_2^2$

$\Delta u = 0$

Def $\quad u$ is harmonic in Ω iff

$u \in C^2(\Omega)$ and $\Delta u(x) = 0 \quad \forall x \in \Omega$

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$U(x)$ = expecting gain starting from x

= probability of exiting

$\Delta U = 0$ in Ω, U is a harmonic function

$\Delta U = u_{x_1x_1} + u_{x_2x_2} + \dots + u_{x_nx_n}$

Boundary conditions

$$\begin{cases} \Delta U = 0 & \text{in } \Omega \\ U = \begin{cases} 1 & \text{on } \Gamma_o \\ 0 & \text{on } \Gamma_c \end{cases} \end{cases} \quad (BVP)$$

$U: \overline{\Omega} \to \mathbb{R}$

defined on the whole closure of Ω

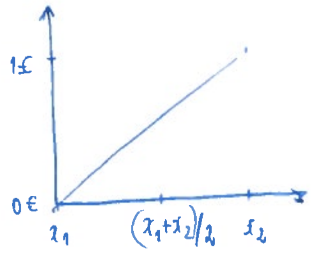
Given $g: \partial\Omega \to \mathbb{R}$

BVP $\begin{cases} \Delta U = 0 & \text{in } \Omega \\ U = g & \text{on } \partial\Omega \end{cases}$ Dirichlet problem

$\boxed{m=1}$

$\boxed{u_{xx}=0}$ $\rightarrow u_x = b = cte$

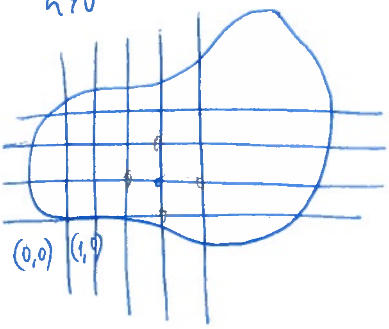
$\rightarrow u(x) = ax+b$



$$u(x) = \frac{x-x_1}{x_2-x_1}$$

$m=2$ $\mathbb{R}^2$

$h>0$



$\boxed{p_{ij} = (ih, jh) \quad i,j \in \mathbb{Z}}$

$\rightarrow$ p_{ij} is interior if has four neighbours belong to Ω $(\in \Omega)$

p_{ij} is a boundary point otherwise

$u^h = u^h(p_{ij})$

$$\Delta_h u^h(p_{ij}) := \frac{1}{h^2}\left\{u(p_{i+1,j}) + u(p_{i-1,j}) + u(p_{i,j+1}) + u(p_{i,j-1}) - 4u(p_{ij})\right\}$$

$$= \frac{4}{h^2}\left\{\text{average of } u \text{ at the 4 neighbours of } p_{ij} - u(p_{ij})\right\} \quad (=0)$$

at every point the value of the point is the average of the 4 neighbours

(DBVP)

$$\begin{cases} \Delta^h u^h = 0, & \forall p_{ij} \text{ interior} \\ u^h = g, & \forall p_{ij} \text{ boundary} \end{cases}$$

<u>Definition</u>

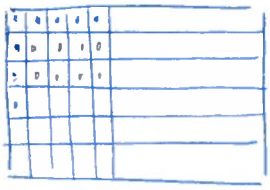
u^h is an ideal image iff $\Delta h\, u^h = 0$, $\forall$ interior pixel

<u>Theorem 1</u>

$\forall \Omega$ bounded open set, $\forall h>0$

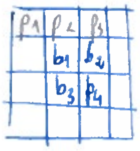
$\exists!$ u^h satisfying (DBVP)

Particular case



$$\begin{cases} \Delta_h u^h = 0 \\ u(\cdot) = g(\cdot) \end{cases}$$

4x4



p → known
b → unknown

$$4b_1 - b_2 - b_3 = \overset{p_2 + p_5}{C_1}$$

$(\dots)$

→ linear system, $A\vec{b} = \vec{c}$, $A = \begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix}$

Symmetric
positive definite (all det are positive)
⇒ $\det A > 0$ ⇒ system invertible ⇒ unique solution

$k_1 \times k_2$?

≈ $k_1 \times k_2$ int pixels, L eqn's

matrix A $(k_1 \times k_2) \times (k_1 \times k_2)$

$$\begin{pmatrix} 4 & & & & & & & \\ & 4 & & & & & & \\ -1 & 0 & 0 & -1 & 4 & -1 & 0 & \cdots & 0 & -1 \\ & & & & & \ddots & & \\ & & & & & & \ddots \end{pmatrix}$$

semi (=)
diagonal dominant ⇒ (sum < diagonal number)

A symmetric (square matrix)
(semi) diagonal ~~determ.~~ dominant

Iterative methods

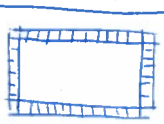
Jacobi or Gauss-Seidel

$A = A_1 + A_2$, $A_1 = 4Id$, $A_2 \begin{pmatrix} 0 & & 0 \\ & \ddots & -1 \end{pmatrix}$

$(A_1 + A_2)\vec{b} = \vec{c}$

$4\vec{b} = \vec{c} - A_2\vec{b}$

$$\boxed{\vec{b}_{k+1} = \frac{\vec{c} - A_2 \vec{b}_k}{4}}$$ iterative method   converges $\forall \vec{b}_k$

⇕  image with given borders

The ideal image computation

- give 0 to all interior points

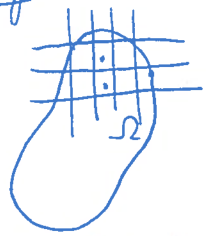
first iteration → (substitution of the average of the 4 neighbours)

<u>Theorem 1</u>

Given ∀Ω bdd open set in $\mathbb{R}^d$, ∀h, ∀values of the boundary pixels

∃! ideal image having them as boundary pixels

<u>Proof</u>



boundary values known ⇒ linear system

X equations and X unknowns ⇒ I only need to prove existence or uniqueness

It is enough by linear algebra to prove <u>uniqueness</u>

↓

2 images w/ the same boundary pixels

$$\begin{cases} \Delta_h u^h = 0 \\ \Delta_h v^h = 0 \end{cases}$$ 2 images

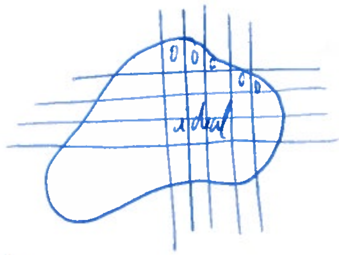
$u^h = v^h$ on boundary points

⇓ ?

$u^h = v^h$ for all int pixels

$w^h := u^h - v^h$

$\boxed{\Delta_h w^h = 0}$ and $\boxed{w^h = 0}$ at every boundary pixels



I need to proof that all the pixels are 0, $w^h = 0$ at interior points

Suppose not, $w^h \neq 0$

⇒ ~~there is~~ the largest pixel has positive value and ~~the largest~~ is achieved at a interior pixel ①

or the smallest pixel (value) is negative and is ~~also~~ achieved at an interior point

① interior point that is the average of the 4 neighbours

largest that is the average of the 4 neighbours ⇒ 4 neighbours are the same and = to the largest value

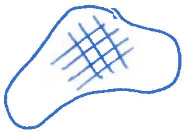
At some point, a boundary pixel will contribute

it is zero ⇒ it can't be the same largest value //

<u>Maximum principle</u>

Ideal image? (Image without edge) (harmonic function + image)

45°/30° degrees lattice? Will I get the Laplace operator?



→ The Laplace operator is invariant under rotations

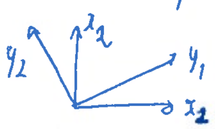
$\Delta u = u_{11} + \dots + u_{nn}$, Hessian matrix $\underset{(n\times n)}{D^2 u} = [u_{ij}] = [\partial x_i x_j u]$

$= \text{trace } D^2_u$

Rotation in $\mathbb{R}^n$ → $y = Ox$, O $n\times n$ orthogonal matrix

↓ orthonormal change of basis in $\mathbb{R}^n$; ↓ $OO^t = O^t O = Id$

$u(x) = \tilde{u}(y) = u(y)$



$D^2_y u = O^t D^2_x u \, O$?

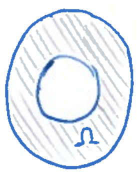
orthogonal (same that inverse)

$\Delta_y u = \text{trace}\left(O^t D^2_x u \, O\right) = \text{trace}\left(O^{-1} D_x^2 u \, O\right) = \text{trace } D^2_x u = \Delta_x u$ □

$n=2$ check that $u_{x_1x_1} + u_{x_2x_2} = u_{y_1y_1} + u_{y_2y_2}$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$ (exercise)

Problem 3 & 4



$r = |x| = \sqrt{x_1^2 + x_2^2} \;(=)\; \sqrt{x_1^2 + \dots x_n^2}$

↓ distance to the origin

$u(x) = u(x_1, \dots, x_n) = \mathbf{u}(r)$ only depends on the radius

u has radial symmetry

Laplace operator acting on radially symmetric functions:

$\partial_{x_i} r = \dfrac{\cancel{2} x_i}{\cancel{2}\sqrt{}} = \dfrac{x_i}{r}$

$u_{x_1} = u_r \, \partial r_{x_1} = u_r \dfrac{x_1}{r}$

↓ ex

$u_{x_1x_1} = (\dots) + (\dots) + (\dots)$

At the end, $\Delta u = u_{rr} + \dfrac{n-1}{r} u_r$, in $\mathbb{R}^n$

& u is a function of 1 real variable $\underline{r \geq 0}$

PDE → ODE

$$\boxed{\Delta U = r^{1-n} \left(r^{n-1} u_r\right)_r}$$

in $\mathbb{R}^2$, $u = u(x_1, x_2)$ not radially symmetric
$= \tilde{u}(r, \theta) \to$ polar coordinates

$$\begin{cases} x_1 = r\cos\theta \\ x_2 = r\sin\theta \end{cases}$$

Exercise

$$\boxed{\Delta v = v_{rr} + \frac{v_r}{r} + \frac{v_{\theta\theta}}{r^2}}$$

in $\mathbb{R}^3$, spherical coordinates

(BVP)

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \subset \mathbb{R}^n, \ n \geq 1 \\ u = g & \text{on } \partial\Omega \end{cases}$$

$g : \partial\Omega \to \mathbb{R}$ given

Theorem 2

If Ω is an open bounded set and g is a continuous function on $\partial\Omega$, there is at most one solution u being continuous up to the boundary in $\bar{\Omega}$ & twice diff in Ω

(uniqueness result) Info: $\exists$ a solution (not a single result)

(Discrete case)

Theorem 3 (Maximum Principle)

Suppose Ω open bounded in $\mathbb{R}^n$,

$u \in C^2(\Omega) \cap C(\bar{\Omega})$ continuous up to boundary, and $\Delta u \geq 0$ in Ω (i.e., u is subharmonic)

2 times diff in Ω (interior of Ω)
and all the derivatives should be continuous

Then

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$$

any continuous function with a compact domain has a max

Proof

Take $\varepsilon > 0$

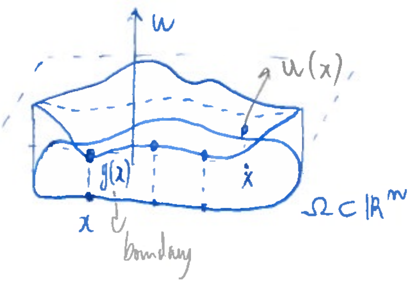
Consider

$u_\varepsilon(x) := u(x) + \varepsilon|x|^2$
$x \in \bar{\Omega} \subset \mathbb{R}^n$

Claim

$$\max_{\bar{\Omega}} v_\varepsilon = \max_{\partial\Omega} u_\varepsilon$$

Proof of Claim



Suppose that the claim is not true and, therefore, the maximum of u_ε in $\bar{\Omega}$ is achieved at an interior point $x_0 \in \Omega$,
i.e.,

$$\exists\, x_0 \in \Omega, \quad u_\varepsilon(x_0) = \max_{\bar{\Omega}} u_\varepsilon$$

$$\partial_{x_1} v_\varepsilon(x_0) = 0$$

$$\vdots$$

$$\partial_{x_n} v_\varepsilon(x_0) = 0$$

$$\left.\begin{array}{l} \partial_{x_1x_1} v_\varepsilon(x_0) \le 0 \\ \vdots \\ \partial_{x_nx_n} v_\varepsilon(x_0) \le 0 \end{array}\right\} \Rightarrow \boxed{\Delta u_\varepsilon(x_0) \le 0}$$

$$\Delta u_\varepsilon(x_0) = \Delta u(x_0) + \Delta\left(\varepsilon|x|^2\right)(x_0) = \Delta u(x_0) + 2\varepsilon n$$

$\downarrow$

$\geq 2\varepsilon n$ → contradiction

Thus

$$\forall\, x \in \Omega, \quad v_\varepsilon(x) \le \max_{\partial\Omega} u_\varepsilon = \max_{y \in \partial\Omega}\left(u(y) + \varepsilon|y|^2\right) \xrightarrow{\varepsilon \to 0} \max_{\partial\Omega} u$$

$$u(x) \le u_\varepsilon(x)$$

$\therefore \boxed{u(x) \le \max_{\partial\Omega} u}$ Maximum principle for subharmonic ~~fcn~~ functions

$\Downarrow$

In the same way, Minimum principle for superharmonic functions ($\Delta u \le 0$ in Ω)

$\Downarrow$

$\Delta u = 0$ in Ω

then

$$\boxed{\max_{\bar{\Omega}} u = \max_{\partial\Omega} u} \quad \& \quad \boxed{\min_{\bar{\Omega}} u = \min_{\partial\Omega} u}$$

~~As a consequence~~

As a consequence of the Maximum Principle is the uniqueness for the (BVP), even for the more general PDE

non-homogeneous equation $\boxed{-\Delta w = f}$ w/ a r.h.s.

where f is a given function in Ω

Proof

$$\begin{cases} -\Delta u = f \\ u|_{\partial\Omega} = g \end{cases} \qquad \begin{cases} -\Delta v = f \\ v|_{\partial\Omega} = g \end{cases}$$

two solutions

$w := u - v$

$\Delta w = 0$ in Ω (is harmonic)

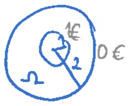
&

$w|_{\partial\Omega} = 0 \Rightarrow \max_{\partial\Omega} w = \min_{\partial\Omega} w = 0 \underset{MP}{\Rightarrow} \boxed{w \equiv 0}$ □ (uniqueness proved)

Corollary 4

The solution of Pb 3

$$\begin{cases} \Delta u = 0 \text{ in } \{1 < |x| < 2\} \subset \mathbb{R}^2 \\ u = \begin{cases} 1 & \text{if } |x| = 1 \\ 0 & \text{if } |x| = 2 \end{cases} \end{cases}$$

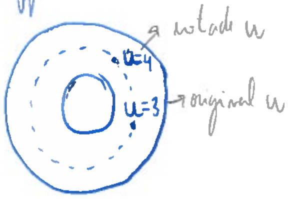


is radially symmetric, $u = u(r)$

MP ⇒ $\boxed{\text{uniqueness} \Rightarrow \text{symmetry properties of BVP}}$

Proof

Suppose

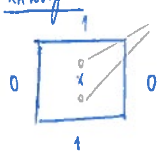


they are both harmonic

they both solve the BVP ⇒ they should be the same by MP ⇒ contradiction

(same BC because of rotational invariance)

Example



I should get the same $u(0,a) = u(0,-a)$, $\forall a \in (0,1)$

u satisfies $u(x_1,x_2) = u(x_1,-x_2)$, $\forall (x_1,x_2) \in (-1,1)\times(-1,1)$

i.e. u is an even function of x_2

Pf

$v(x_1,x_2) := u(x_1,-x_2)$

they solve the same problem, by uniqueness $\underline{u = v}$ □

⇓

symmetry

$v_{x_1x_1} = u_{x_1x_1}$

$v_{x_2} = -u_{x_2}(x_1,-x_2)$, $v_{x_2x_2} = u_{x_2x_2}(x_1,-x_2)$

$\Delta v = 0 \quad v|_{\partial Q} = u|_{\partial Q}$