Math Modeling in Hydrodynamics

André Nachbin
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$\odot_1$ Physical model
$\odot_2$ Math model } Choices
$\odot_3$ Numerical model

$$\frac{dy(t)}{dt} = \lambda y, \quad y(0) = y_0$$

Exp. Euler: $y^{n+1} = y^n + \Delta t \lambda y^n$

Imp. Euler: $y^{n+1} = y^n + \Delta t \lambda y^{n+1}$

Trap. Rule: $y^{n+1} = y^n + \frac{\Delta t \lambda}{2}(y^n + y^{n+1})$

$Re(\lambda) < 0$ = Test problem (Absolute stability)

$\lambda = i\omega$

Get our first PDE

Conservation of mass

$\rho(\vec{x}, t)$ = fluid density = specific mass = $\frac{[\text{mass}]}{[\text{volume}]}$

w = arbitrary control region

Ω $\quad m(t, w) = \int_w \rho(\vec{x}, t)\, dV \qquad dV = dx\,dy\,dz$

$$\frac{dm}{dt} = \frac{d}{dt}\int_w \rho(\vec{x}, t)\, dV = -\int_{\partial w} \rho\, \vec{u}\cdot\vec{n}\, dA$$

mass balance eq.

$$\int_w \frac{\partial \rho}{\partial t}(\vec{x}, t)\, dV = -\overbrace{\int_w \nabla\cdot(\rho\vec{u})\, dV}^{\text{Divergence theorem}}$$

$$\int_w [\rho_t + \nabla\cdot(\rho\vec{u})]\, dV = 0$$

Conclude: $\rho_t + \nabla\cdot(\rho\vec{u}) = 0 \qquad \nabla = (\partial/\partial x, \partial/\partial y)$

$\vec{u}(x,t)$

$$\vec{x}(t) = \begin{bmatrix} x(t, \vec{a}) \\ y(t, \vec{a}) \\ z(t, \vec{a}) \end{bmatrix}$$

position of fluid particle in time that started at position $\vec{a}$

$$\frac{d\vec{x}}{dt}(t; \vec{a}) = \vec{u}(x(t; \vec{a}), t)$$

velocity following particle $\vec{a}$

$$\frac{d^2 x_i}{dt^2} = \left[\frac{\partial u_i}{\partial t} + \frac{\partial u_i}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u_i}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}\right]$$

$$\frac{d^2 x_i}{dt^2} = \underbrace{\frac{D}{Dt}}_{} u_i = \left[\partial/\partial t + \vec{u}\cdot\nabla\right] u_i$$

Material derivative transport.

$\rho_t + \nabla\cdot(\rho\vec{u}) = 0$

$\rightarrow \frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla\cdot\vec{u}$

$\nabla\cdot\vec{u} = 0$ flow incompressible

$\nabla\cdot\vec{u} > 0$ 

2D: $u_x + u_y = 0$