Math Modeling in Hydraul

① Physical model
② Math model
③ Numerical model

choices

water is incompressible

$$\frac{dy(t)}{dt} = \lambda y, \quad y(0) = y_0$$

explicit Euler: $y^{n+1} + \Delta t \lambda y^n$

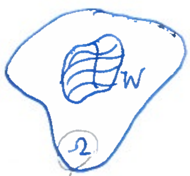
implicit Euler: $y^{n+1} = y^n + \Delta t \lambda y^{n+1}$

Trap rule: $y^{n+1} = y^n + \frac{\Delta t}{2} \lambda (y^n + y^{n+1})$

If $Re(\lambda) < 0$: test problem (absolute stability)
solution should decay in time

If $\lambda = i\omega$, the solution is oscillatory

Get our first PDE
Conservation of mass
$\rho(\vec{x}, t)$ = density = specific mass = [mass] / [volume]



fluid domain

W = arbitrary control region

$$m(t; W) = \int_W \rho(\vec{x}, t)\, dV \qquad dV = dx\,dy\,dz$$

flow of mass

$$\frac{dm}{dt} = \frac{d}{dt}\int_W \rho(\vec{x}, t)\, dV = -\int_{\partial W} \rho \vec{u} \cdot \vec{n}\, dA$$

surface integral

mass balance eq.

mass balance

Divergence theorem

$$\int_W \frac{\partial \rho}{\partial t}(\vec{x}, t)\, dV = -\int_W \nabla \cdot (\rho \vec{u})\, dV$$

$$\int_W [\rho_t + \nabla \cdot (\rho \vec{u})]\, dV = 0$$

for continuity

Conclude $\boxed{\rho_t + \nabla \cdot (\rho \vec{u}) = 0}$ $(\vec{x}, t)$, $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$

$\vec{X}(t)\ \begin{bmatrix} x(t,\hat{a}) \\ y(t,\hat{a}) \\ z(t,\hat{a}) \end{bmatrix}$

$\downarrow$

position of a fluid particle in time that started at position $\hat{a}$

$$\frac{d\hat{X}}{dt}(t;\hat{a}) = \vec{u}(x(t;\hat{a}),t)$$

velocity following particle $\vec{a}$

grad of the particle in its moving frame

the acceleration => compute time derivative on the moving frame (lagrangian way)

$$\frac{d^2 x_i}{dt} = \left[\frac{\partial u_i}{\partial t} + \frac{\partial u_i}{\partial x}\frac{dx}{dt} + \frac{\partial u_i}{\partial y}\frac{dy}{dt} + \frac{\partial u_i}{\partial z}\frac{dz}{dt}\right]$$

$$\frac{d^2 x_i}{dt^2} = \underbrace{\frac{D\,u_i}{Dt} = \left[\frac{\partial}{\partial t} + \vec{U}\cdot\nabla\right]}_{\text{material Derivative/transport}} u_i$$

$$\begin{cases} \rho_t + \nabla\cdot(\rho\vec{u}) = 0 \\ \dfrac{1}{\rho}\dfrac{D\rho}{Dt} = -\nabla\cdot\vec{u} \end{cases}$$

$\boxed{\nabla\cdot\vec{u}=0}$ => flow ~~fluid~~ incompressible => $\frac{D\rho}{Dt}=0$ => density is not changing

$\nabla\cdot\vec{u}>0$ fluid diverging

$\nabla\cdot\vec{u}<0$

(2D) $u\vec{i}+v\vec{j}=0$