Nonlinear - 2

$$\frac{d(\vec{x})}{dt} = \vec{u}(\vec{x}(t,a), t)$$

$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{u}\cdot\nabla$. Then the equation writes, at 1D,

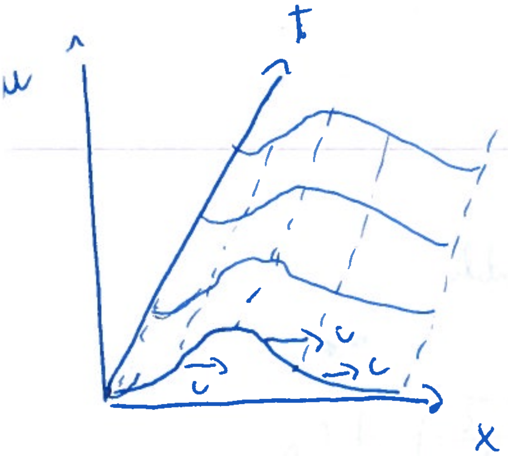
(a) $\frac{D}{Dt} = \partial_t + c\,\partial_x \overset{=0}{}$ if $u \equiv c$ is const  or (b) $\frac{D}{Dt} = \partial_t + u\,\partial_x \overset{=0}{}$ in general.

Linear transport  Transport equation

(a) $u_t + c u_x = 0$ ⇝ solution $u(x,t) = f(x-ct)$

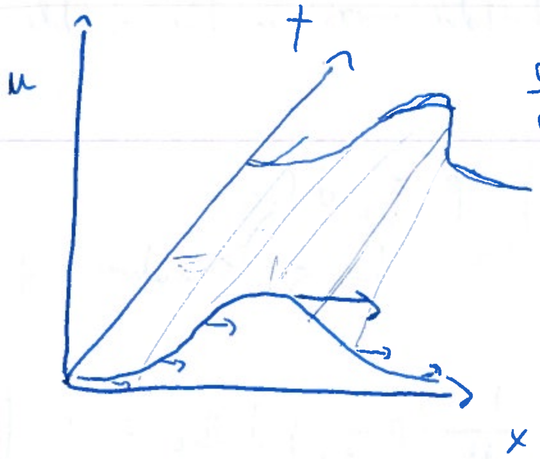
(b) $u_t + u u_x = 0$ ⇝ solution implicit $u(x,t) = f(x-ut)$

(a)



$\frac{dx}{dt} = c$ const

(b)



$\frac{dx}{dt} = u$

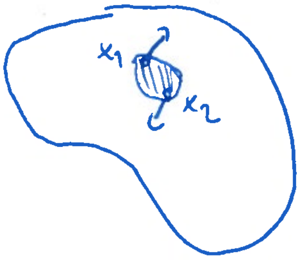
"Every point is being transported with velocity $c$"

"Speed of points is bigger when the wave is higher".

Ex: $u_t + c u_x = 0$ with initial condition $u(x,0) = e^{ikx} = \cos(kx) + i\sin(kx)$

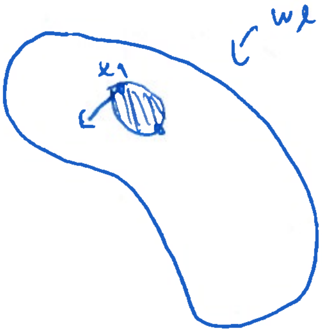
Then $u(x,t) = a(t)\, e^{ikx}$. Plugging we find that $\frac{da}{dt} = -ikca = -i\omega a$, therefore ($\omega = kc$)
$a(t) = Ae^{-i\omega t}$, therefore $u(x,t) = Ae^{(ikx - i\omega t)}$

Eulerian | Lagrangian




$t=0$


$t=0$




we fix the domain $\Omega$

$t=1$

$\Omega(t)$

$t=1$

Here we follow the points moving.

## Conservation of mass – Lagrangian

$$= \frac{d}{dt}\left(m(t,\Omega(t))\right) = \frac{d}{dt}\int_{\Omega(t)} \rho(\vec{x},t)\cdot d\Omega(t) = 0$$

total mass in the variable domain $\Omega(t)$

$\Omega(t) = \Omega(\Omega_0, t)$

$$= \frac{d}{dt}\int_{\Omega_0} \rho\, J_t\, d\Omega_0$$

$J$ = Jacobian of change of variables

$$= \int_{\Omega_0} \frac{d}{dt}(\rho J_t)\, d\Omega_0 = \int_{\Omega_0}\left(\left(\frac{d}{dt}\rho\right)J + \rho\frac{dJ}{dt}\right) d\Omega_0$$

$$\frac{dJ}{dt} = (\nabla\cdot\vec{u})\,J$$

$$= \int_{\Omega(t)}\left[\frac{d\rho}{dt} + \rho(\nabla\cdot\vec{u})\right] d\Omega(t)$$

and again we get

$$0 = \frac{d\rho}{dt} + \rho(\nabla\cdot\vec{u})$$

## Euler Equations

$$\begin{cases} \rho \dfrac{D\vec{u}}{Dt} = -\nabla p + \rho \vec{b} \\[2ex] \dfrac{1}{\rho}\dfrac{D\rho}{Dt} = -\nabla\cdot\vec{u} \end{cases}$$

we suppose now it's incompressible and irrotational:

$\nabla\cdot\vec{u} = 0$ and $\nabla\times\vec{u} = 0$ $\qquad \vec{u} = (u, v, 0)$

$$u_x = -v_y$$
$$u_y = v_x$$

← like Cauchy equations

$$\frac{df}{dz} \leftarrow \frac{f(z+\Delta t) - f(z)}{\Delta t}$$

(complex variable) gives Cauchy-Riemann eqs:

$$f(x,y) = \phi(x,y) + i\psi(x,y)$$

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$$

## Potential Theory

$\Phi(x,y) = \phi(x,y) + i\psi(x,y)$ where $\phi(x,y)$ is velocity potential, $(u,v) = \nabla\phi$. Then

$$\frac{d\Phi}{dt} = u - iv$$

← $\Phi$ is analytic if $\vec{u}$ is incompressible and irrotational.