$$\beta \phi_{xx} + \phi_{yy} = 0$$

$$\left.\begin{aligned} &\eta_{tt} + \alpha \phi_x \eta_x - \frac{1}{\beta}\phi_y = 0 \\ &\eta + \phi_t + \frac{1}{2}\alpha \phi_x^2 + \frac{1}{2}\frac{\alpha}{\beta}\phi_y^2 = 0 \end{aligned}\right\} \circledast_1$$

$\circledast_1$ @ $y = 1+\alpha\eta(x,t)$

$\circledast_2$ @ $y = 0$

$\circledast_2$ $\frac{d\phi}{dn} = \phi_y = 0$

$$\alpha = \frac{a}{h_0}, \quad \beta = \frac{h_0^2}{\lambda^2} \text{ (disp)}$$

$$\boxed{\phi(x,y,t) = \sum_{n=0}^{\infty} (-\beta)^n \frac{y^{2n}}{(2n)!}\partial_x^{2n} f(x,t)}$$

$(\circledast_1 + \circledast_2)$

$$\eta_t + \alpha f_x \eta_x + (1+\alpha\eta) f_{xx} - \frac{\beta}{6} f_{xxxy} = \mathcal{O}(\alpha^2, \alpha\beta, \beta^2)$$

$$\eta + f_t - \frac{\beta}{2} f_{xxt} - \frac{\alpha}{2} f_x^2 = \underbrace{\mathcal{O}(\alpha^2, \alpha\beta, \beta^2)}_{\text{higher order}}$$

Weakly non linear: $\alpha = \mathcal{O}(\epsilon)$

Weakly dispersive: $\beta = \mathcal{O}(\epsilon)$

Truncate: $\tilde{u} = f_x$ ← (Speed at the bottom)

$$\begin{cases} \eta_t + \alpha \tilde{u}\eta_x + (1+\alpha\eta)\tilde{u}_x - \frac{\beta}{6}\tilde{u}_{xxx} = 0 \\ \eta_x + \tilde{u}_t - \frac{\beta}{2}\tilde{u}_{xxt} + \frac{\alpha}{2}(\tilde{u}^2)_x = 0 \end{cases}$$

$$\mathcal{U} = \frac{1}{Y}\int_0^Y \phi_x(x,y,t)\,dy = \text{(average speed)}$$

$$Y = 1+\alpha\eta(x,t) \quad \Big| = \tilde{u}(x,t) - \frac{\beta}{6}(1+\alpha\eta)^2 \tilde{u}_{xx} + \mathcal{O}(\beta^2)$$

$$\tilde{u}(x,t) = \mathcal{U}(x,t) + \frac{\beta}{6}\mathcal{U}_{xx}(x,t) + \text{higher order terms.}$$

$$\eta_t = -\mathcal{U}_x + \mathcal{O}(\alpha)$$

We can change time derivatives to Spatial derivatives.

Dimensionless Boussinesq equations

$$\begin{cases}\eta_t + [(1+\alpha\eta)\mathcal{U}] = 0 \\ \mathcal{U}_t + \eta_x + \alpha\,\mathcal{U}\,\mathcal{U}_x + \frac{\beta}{3}\eta_{xtt} = 0\end{cases}$$

If $\alpha = 0$ and $\beta = 0$

$$\begin{cases}\eta_t + u_x = 0 \\ u_t + \eta_x = 0\end{cases}$$

$u(x,0) = \eta(x,0) = f(x)$ everything is moving to the right.

$\Rightarrow \eta_t + \eta_x = 0 \Rightarrow \eta = f(x-t)$

ansatz

$$\tilde{u} = \eta + \alpha A(x,t) + \beta B(x,t) + \mathcal{O}(\alpha^2, \alpha\beta, \beta^2)$$

$A = ?$ $B = ?$

nonlinear effect

dispersive Term

$$\boxed{\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{\beta}{6}\eta_{xxx} = 0}$$ KdV

Traveling wave

Traveling speed related ~~with~~ with amplitude

$\eta_t + \eta_x = 0$

$\eta(x,0) = e^{ikx}$, $k = \frac{2\pi}{\lambda}$

$\eta(x,t) = a(t)e^{ikx}$

$e^{ik(x - \underbrace{\frac{\omega}{k}}_{=1} t)}$ Fourier Mode

$\omega(k) = k$

$\boxed{\Omega(k) = k}$

~~Total~~

$\frac{\partial \omega}{\partial k}$ = group velocity

$\frac{\omega}{k}$ ← phase velocity

Boussinesq equation in the linear regime ($\alpha = 0$)

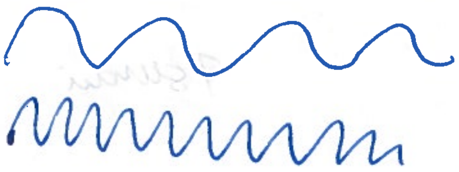
$$\begin{cases} \eta_t + u_x = 0 \\ u_t + \eta_x + \frac{\beta}{3}\eta_{xtt} = 0 \end{cases}, \qquad \eta = A e^{i(kx - \omega t)}, \quad u = B e^{i(kx - \omega t)} \quad \text{at time } t$$

Exercise

$$\underbrace{\begin{bmatrix} * & * \\ * & * \end{bmatrix}}_{\det = 0} \begin{bmatrix} A \\ B \end{bmatrix} = \Big[$$

$$\omega = \omega(k) \neq \text{linear in } K = \frac{P(k)}{q(k)} \quad \text{(rational function)}$$

$$\eta = A e^{ik\left(x - \frac{\omega(k)}{k} t\right)}$$



the waves travel at different speeds

We can get dispersion relations in numerical methods.

$$\boxed{\eta_t + \eta_x = D(\Delta t, \Delta x)\,\eta_{xx}}$$

$$\begin{cases} \phi_{xx} + \phi_{yy} = 0 \\ \phi_{tt} + g\phi_y = 0 \quad y = 0 \\ \phi_y = 0 \quad y = -h_0 \end{cases}$$

eliminated η $\quad \eta = -\phi_t$

$$\phi(x, y, t) = F(y)\,G(t)\,e^{ikx}$$ solve ~~vertical~~ vertical ODE in $F(y)$

$$\underbrace{\cosh(k(y + h_0))}\, e^{ikx}$$

harmonic extension of the Fourier mode

$$\boxed{\omega^2(k) = Kg\tanh^2(kh_0)}$$

Full dispersion relation

$\boxed{\text{Assymptotic of Integrals}}$

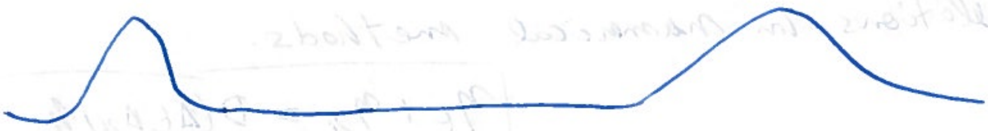
$\omega^2(k) = kg \tanh(kh_0)$

Shallow water regime: $h_0 \to 0$

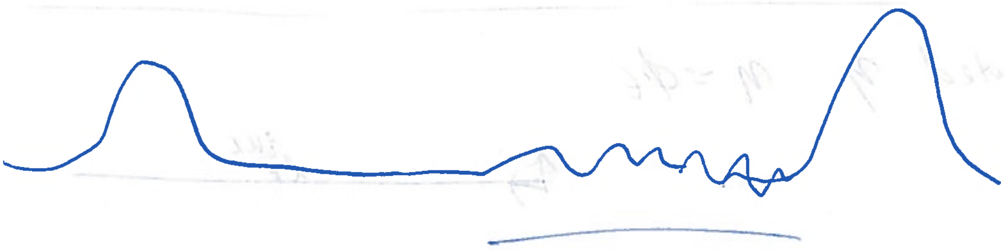
$\omega^2(k) \sim k^2 g h_0$

$\frac{\omega(k)}{k} = \sqrt{g h_0}$ $g \simeq 10\, m/s^2$ $h_0 \sim 4000\, m$

Tsunami 720 km/h



Linear Problem



Nonlinear Problem

$\sim Ai(x-t)$

Airy function

higher frequencies falling behind.