Linearize equations and we get

- Mass conservation : $\eta_t + \cancel{g}H u_x = 0$
- Momentum conservation : $\cancel{H} u_t + g\cancel{H} \eta_x = 0$

$$\begin{cases} \eta_t + \cancel{g}H u_x = 0 \\ u_t + g\eta_x = 0 \end{cases}$$

$$\Downarrow$$

$$\eta_{tt} - \underbrace{gH}_{c^2} \eta_{xx} = 0$$

Wave equation.

Speed of Tsunami

$$c = \sqrt{g\underset{\text{3km deep}}{H}} \approx \sqrt{10 \times 4000} = 200 \, m s^{-1}$$

$$\approx 720 \, km \, h^{-1}$$

Back to non-linear equations. Developing momentum conservation

$$\cancel{h}u_t + h u u_x + g h h_x = 0 \Longrightarrow u_t + u u_x + g h_x = 0$$

$$\underbrace{+(hu)_x + h_t}_{=0 \text{ By mass conservation}}$$

we forget about shocks for a while

$$\begin{pmatrix} h \\ u \end{pmatrix}_t + \underset{A}{\begin{pmatrix} u & h \\ g & u \end{pmatrix}} \begin{pmatrix} h \\ u \end{pmatrix}_x = 0$$

In the linear case the matrix is $\begin{pmatrix} 0 & H \\ g & 0 \end{pmatrix}$.

We're going to pick vector $(l_1 \ \ l_2)$ and consider

$$(l_1 \ \ l_2) \begin{pmatrix} h \\ u \end{pmatrix}_x + (l_1 \ \ l_2) A \begin{pmatrix} h \\ u \end{pmatrix}_x = 0$$

We will chose (l_1, l_2) as left eigenvector of A.

Nonlinear -3

Complex potential $\Phi(z) = \underset{\text{velocity potential}}{\phi(x,y)} + i \underset{\text{stream function}}{\psi(x,y)}$.

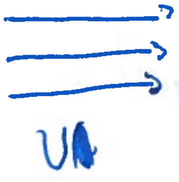
$(u,v) = \nabla\phi$

$$\frac{\partial\Phi}{\partial z} = \frac{\partial\Phi}{\partial x} = -i\frac{\partial\Phi}{\partial y} = \phi_x + i\psi_x = -i(\phi_y + i\psi_y) = u - iv$$

$(-v,u) = \nabla\psi$

$$\frac{d\Phi}{dz} = u - iv = \frac{dx}{dt} - i\frac{dy}{dt} = \frac{d\bar z}{dt}$$

Ex1:



$\rightsquigarrow \Phi = Uz$

complex potential

↓

Ex2:

↑

Milne Thompson circle theorem: Let $f(z)$ analytic in $|z|<R$. Then in the presence of the cilinder the relative potential is

$$\Phi(z) = f(z) + \overline{f\left(\frac{R^2}{\bar z}\right)}.$$

$$\overline{\left(\frac{R^2}{\bar z}\right)} = \frac{R^2}{z}$$

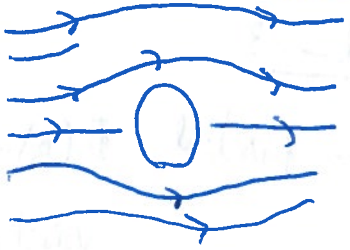
Essentially we have no conditions on the boundary of the problem.

$$\Phi(Re^{i\theta}) = 2\,\mathrm{Re}\, f(x) \in \mathbb{R} \Rightarrow \psi \equiv 0$$

Ex: $\psi(x,y) =$ const are tangent to the velocity vector

$z^* = R^2/\bar z$ image point of z

level sets of ψ
stream lines.



Using Milne Thompson Th in ex2,

$$\Phi(z) = U\left(z + \frac{R^2}{z}\right) \Rightarrow \psi(x,y) = Uy\left(1 - \frac{R^2}{x^2+y^2}\right)$$

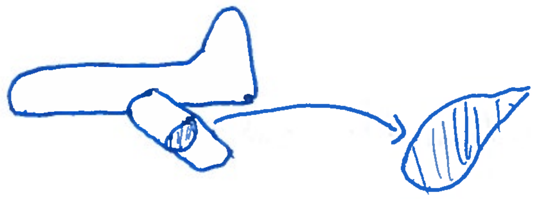
$z = \frac{1}{2}\left(w + \frac{\lambda^2}{w}\right)$ → Joukowsky transformation

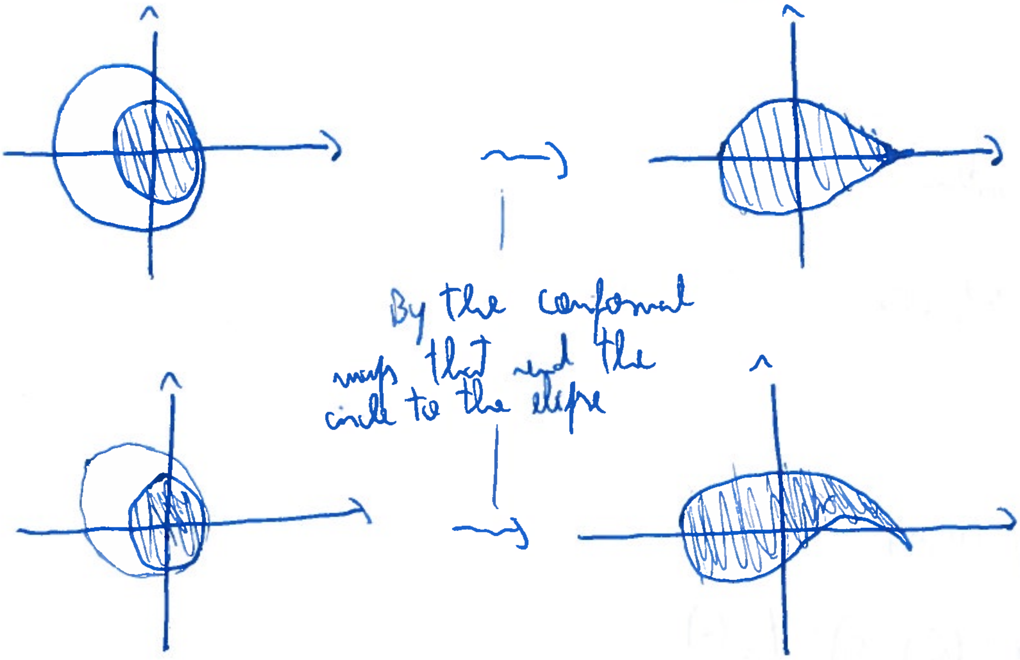
Maps a circle to an elipse.

By chosing λ^2 accordingly you can make the elipse colapse.

<u>Ref</u>: Ablowitz & Fokas, Cambridge University Press.

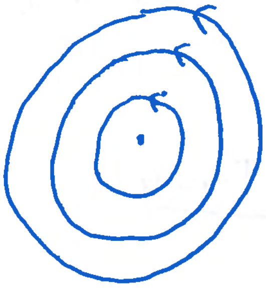
Airplane





By the conformal maps that and the circle to the elipse

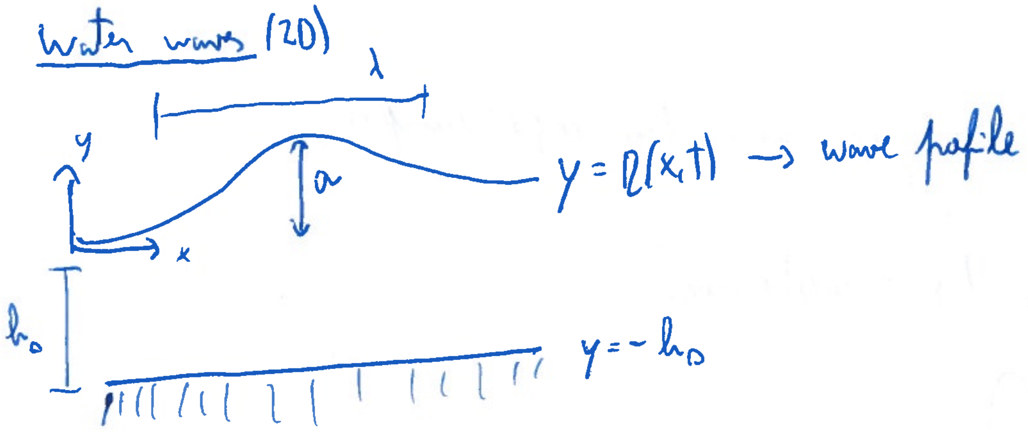
Ex 3: $\Phi(z) = -i\nu \log z$



Wing of airplane

Nachlan 4

Water waves (2D)



Hyp: inviscid, incompressible, irrotational, impermeable soil, etc.

Eq:
$$\begin{cases} \Delta \phi = 0 \\ \dfrac{d\phi}{dn} = 0 \quad \text{in the bottom} \\ \phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) + g\eta = 0 \\ \eta_t + \phi_x \eta_x - \phi_y = 0 \end{cases}$$

$\phi = \phi(x,y,t)$ vel potential, $\nabla\phi = (u,v)$

Initial conditions:
$$\begin{cases} \phi(x, \eta(x,0), 0) = \phi_0(x) \\ \eta(x,0) = \eta_0(x) \end{cases}$$

Nonlinear Potential Theory

Whitham - Linear and Nonlinear Waves.

$c_0 = \sqrt{g h}$

New variables: $x = \lambda \tilde{x}$, $y = h_0 \tilde{y}$, $t = \frac{\lambda}{c_0}\tilde{t}$, $\eta = a\tilde{\eta}$, $\phi = \frac{g\lambda a}{c_0}\tilde{\phi}$

non-dimensional.

Dimensionless equations (drop the ~)

1) $\beta\phi_{xx}+\phi_{yy}=0$ ← "Laplace" eq.

$$\left.\begin{array}{ll}) & \eta_t+\alpha\phi_x\eta_x-\frac{1}{\beta}\phi_y=0 \\ (*) & \\ 3) & \eta+\phi_x+\frac{1}{2}\alpha\phi_x^2+\frac{1}{2}\frac{\alpha}{\beta}\phi_y^2=0 \end{array}\right\}$$

(*) at $y=$ [illegible] $\alpha\eta$

(4) $\frac{d\phi}{dy}=0$, $y=-1$ at ~~$y=1+\alpha\eta$~~

where $\alpha=\frac{a}{h_0}$, $\beta=\frac{h_0^2}{\lambda^2}$.

If $\alpha\to 0$ we get the linearized problem.

α – Non-linearity parameter.

β – dispersive parameter.

Regime $\alpha=O(\varepsilon)$, $\beta=O(\varepsilon)$, $\varepsilon\ll 1$.

weakly nonlinear, weakly dispersive regime.

Ansatz: ☒$_1$ $\phi(x,y,t)=\sum_m y^m f_m(x,t)$ ← Separation of variables.

Plug ☒$_1$ in "Laplace eq." (1) and impose Neumann condition (4)

we get

$$\phi(x,y,t)=\sum_{m=0}^{\infty}(-\beta)^m\frac{y^{2m}}{(2m)!}\partial_x^{2m}f(x,t) \quad \text{where}$$

$f=f_0$

Now we substitute this expression for ϕ in (2), (3)

↑ Non-linear.

Leading order:

$$\phi_x = f_x - \frac{\beta}{2}(1+\alpha\eta)^2 f_{xxx} + \overbrace{O(\beta^2)}^{O(\varepsilon^2)}$$

$$\phi_y = -\beta(1+\alpha\eta) f_{xx} + \underbrace{O(\beta^2)}_{O(\varepsilon^2)}$$

we only want to keep linear terms in α and β. I.e truncate stuff beyond $O(\alpha^2, \alpha\beta, \beta^2)$.

Put this in (2) and (3) and truncate

$$\begin{cases} (2) & \eta_t + \alpha f_x \eta_x + (1+\alpha\eta) f_{xx} - \frac{\beta}{6} f_{xxxx} = 0 \\ (3) & \eta + f_t + \frac{\beta}{2} f_{xxt} + \frac{\alpha}{2} f_x^2 = 0 \end{cases}$$

Boussinesq equations: η and $u \sim$ average of ϕ over y.

$$\frac{1}{Y}\int_0^Y \phi_x(x,y,t)\,dy \equiv U(x,t)$$

Take -4 -

$$\begin{cases} h_t + (hu)_x = 0 \\ u_t + u u_x + h x = 0 \end{cases} \Leftrightarrow \begin{pmatrix} h \\ u \end{pmatrix}_t + \begin{pmatrix} u & h \\ 1 & u \end{pmatrix} \overset{U}{\begin{pmatrix} h \\ u \end{pmatrix}}_x = 0$$

In the linearized system:

$$\begin{cases} h_t + u_x = 0 \\ u_t + h_x = 0 \end{cases} \Rightarrow \begin{cases} (h+u)_t + (\overbrace{h+u}^{R})_x = 0 \\ (\underbrace{h-u}_{L})_t - (h-u)_x = 0 \end{cases}$$

Leading order:

$$\phi_x = f_x - \frac{\beta}{2}(1+\alpha\eta)^2 f_{xxx} + \overset{\overset{O(\varepsilon^2)}{\frown}}{O(\beta^2)}$$

$$\phi_y = -\beta(1+\alpha\eta) f_{xx} + \underbrace{O(\beta^2)}_{O(\varepsilon^2)}$$

we only want to keep linear terms in α and β. I.e truncate stuff beyond $O(\alpha^2, \alpha\beta, \beta^2)$.

Put this in (2) and (3) and truncate

$$\begin{cases} \eta_t + \alpha f_x \eta_x + (1+\alpha\eta) f_{xx} - \frac{\beta}{6} f_{xxxx} = 0 & (2) \\ \eta + f_t + \frac{\beta}{2} f_{xxt} + \frac{\alpha}{2} f_x^2 = 0 & (3) \end{cases}$$

Boussinesq equations: η and $u \sim$ average of ϕ over y.

$$\frac{1}{Y}\int_0^Y \phi_x(x,y,t)\,dy \equiv U(x,t)$$

Talk - 4 -

$$\begin{cases} h_t + (hu)_x = 0 \\ u_t + uu_x + h_x = 0 \end{cases} \Leftrightarrow \begin{pmatrix} h \\ u \end{pmatrix}_t + \begin{pmatrix} u & h \\ 1 & u \end{pmatrix} \overset{U}{\overset{\shortparallel}{\begin{pmatrix} h \\ u \end{pmatrix}}}_x = 0$$

In the linearized system:

$$\begin{cases} h_t + u_x = 0 \\ u_t + h_x = 0 \end{cases} \Rightarrow \begin{cases} (h+u)_t + (\overbrace{h+u}^{R})_x = 0 \\ (\underbrace{h-u}_{L})_t - (h-u)_x = 0 \end{cases}$$