MATHEMATICAL MODELING in HYDRODYNAMICS

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- Φ_1 Physical modeling
- Φ_2 Mathematical modeling — this course
- Φ_3 Numerical modeling.

CHOICES! (one of the most important words in life!)

Φ_1 : Fluid: viscous or inviscid (called ideal fluid).

Flow compressible or incompressible.

Flow irrotational or rotational.

How much Physics we need?

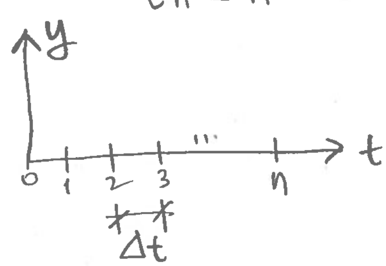
— thermal considerations?

— chemical reaction?

etc...

all legitimate questions.

Φ_3 $$\boxed{\frac{dy}{dt} = \lambda y, \quad y(0) = y_0}$$, $y^n \approx y(t_n)$, $t_n = n\Delta t$



Explicit Euler: $y^{n+1} = y^n + \Delta t \lambda y^n$

Implicit Euler: $y^{n+1} = y^n + \Delta t \lambda y^{n+1}$

Trapezoid Rule: $y^{n+1} = y^n + \frac{\Delta t}{2} \lambda \left(y^{n+1} + y^n\right)$

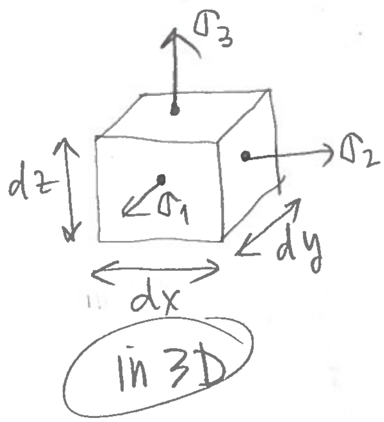
— 3 legitimate choices for our Numerical Model!

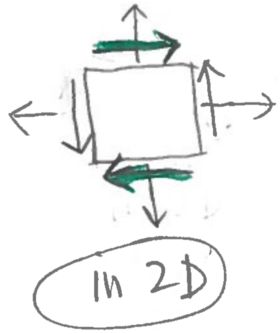
— which one should we choose for an oscillatory problem?

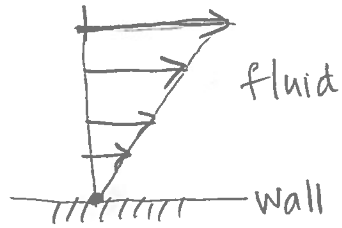
$$\frac{dy}{dt} = i\omega y, \quad \lambda = i\omega \Rightarrow y(t) = y_0 e^{i\omega t}$$

ϕ_2 Math Model:

inviscid fluid — no friction, no tangential stresses
only normal stresses $\Rightarrow$ only pressure
(negative normal stress)
compression x traction.







shear flow
relative motion

$\vec{u} = (u(x,y), 0)$

vorticity $\vec{\omega} = \nabla \times \vec{u} = (v_x - u_y)\vec{k} = -u_y$

$u_y = \frac{\partial u(x,y)}{\partial y}$ partial derivative.

in this case only depends on y!

Notation: partial derivative

$\frac{\partial u(x,y)}{\partial x} = u_x(x,y) =$

$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}$; y = fixed

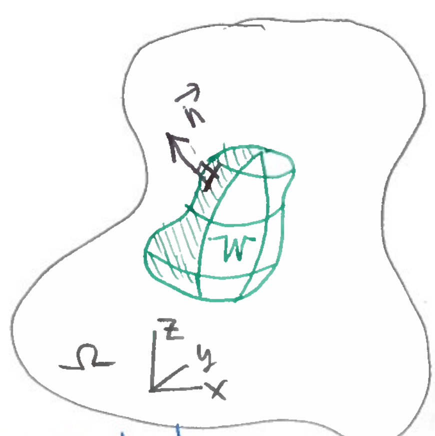
Lets get (deduce) our first PDE:

$\boxed{\text{Conservation of MASS.}}$

let $\underbrace{\rho(\vec{x},t)}_{\text{scalar func.}} \equiv$ density of the fluid $=$ specific mass $= \frac{[\text{Mass}]}{[\text{Volume}]}$, $[\cdot] \equiv$ units

total mass in subdomain $\mathcal{W}$ $\underset{\text{define as}}{\overset{\text{denote by}}{\equiv}}$ $\boxed{m(t;\mathcal{W}) = \int_{\mathcal{W}} \rho(\vec{x},t)\, dV}$

$\iiint_{\mathcal{W}} (-)\, dx\,dy\,dz$.



$\mathcal{W}$ is arbitrary
think as FIXED CONTROL region inside the FLUID domain Ω.

Mass balance in the control subdomain $\mathcal{W}$:

why?

$$\frac{dm}{dt} = \frac{d}{dt}\int_{\mathcal{W}} \rho(\vec{x},t)\,dV = - \underbrace{\int_{\partial\mathcal{W}} \rho\,\vec{u}\cdot\vec{n}\,dA}_{\text{mass flux through the boundary of } \mathcal{W}}$$

surface area

$\boxed{\text{Checking units:}}$

$$\frac{1}{[T]}\,\frac{[M]}{[L]^3}\,[L]^3 = \frac{[M]}{[L]^3}\,\frac{[L]}{[T]}\,[L]^2$$

$$\frac{[M]}{[T]} = \frac{[M]}{[T]}$$

$$\int_{\mathcal{W}} \frac{\partial\rho}{\partial t}(\vec{x},t)\,dV = -\int_{\mathcal{W}} \nabla\cdot(\rho\vec{u})\,dV \quad \text{(Diverg. Thm)}$$

$$\boxed{\int_{\mathcal{W}}\left[\rho_t + \nabla\cdot(\rho\vec{u})\right]dV = 0}\ , \text{ for all } \mathcal{W}\subset\Omega$$

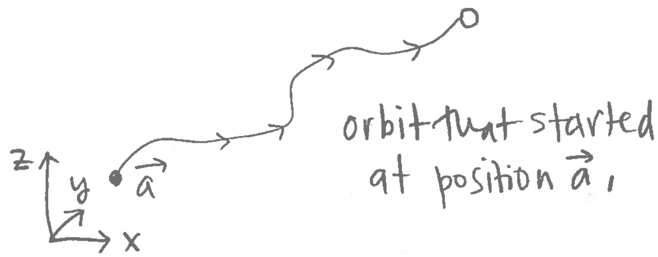
$$\Rightarrow \boxed{\rho_t + \nabla\cdot(\rho\vec{u}) = 0} \quad \text{Conservation of Mass.}$$

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0\ , \quad \boxed{\vec{u} = (u,v,w)}$$

If $\vec{x}$ is the position of a fluid particle what is its acceleration?

$$\vec{x} = \begin{bmatrix} x(t;\vec{a}) \\ y(t;\vec{a}) \\ z(t;\vec{a}) \end{bmatrix}$$

ODE, orbit-like notation



particle velocity $\quad \dfrac{d\vec{x}(t;\vec{a})}{dt} = \vec{u}(\vec{x}(t;\vec{a}),t)$, $\vec{x}(0) = \vec{a}$

velocity of particle that started at $\vec{a}$

particle acceleration following the particle

$$\boxed{\frac{d^2\vec{x}}{dt^2} = \frac{d}{dt}\vec{u}(\vec{x}(t;\vec{a}),t)}$$

Why is this different from $\dfrac{\partial \vec{u}}{\partial t}$?

$$\frac{d^2 x_i}{dt^2} = \frac{\partial u_i}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial u_i}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial u_i}{\partial x_3}\frac{dx_3}{dt} + \frac{\partial u_i}{\partial t}$$

$$= \left[\frac{\partial}{\partial t} + u_1\frac{\partial}{\partial x_1} + u_2\frac{\partial}{\partial x_2} + u_3\frac{\partial}{\partial x_3}\right]u_i$$

$$= \left[\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right]u_i = \boxed{\frac{D u_i}{Dt}}$$

Not an ordinary derivatives – is a partial diff. operator.

$\dfrac{D}{Dt} \equiv$ material derivative or transport derivative (transport performed by $\vec{u}$)

$dx/dt = c$

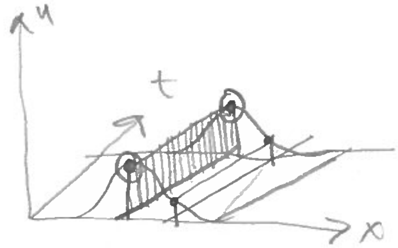
wave eqn. $\quad u_t + c u_x = 0 \Leftrightarrow \dfrac{Du}{Dt} = 0$, $\dfrac{D}{Dt} = \partial_t + c\,\partial_x$

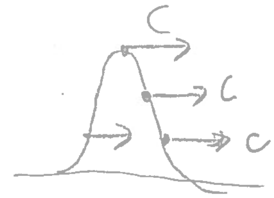
Burger's eqn. $\quad u_t + u u_x = 0 \Leftrightarrow \dfrac{Du}{Dt} = 0$, $\dfrac{D}{Dt} = \partial_t + u\,\partial_x$

1P

(1) $u_t + c u_x = 0$

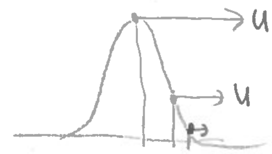
$u(x,0) = f(x)$





(2) $u_t + u u_x = 0$

$u(x,0) = f(x)$



Both $\frac{Du}{Dt} = 0$, but $\frac{D}{Dt}$ are different.

(1) $u(x,t) = f(x-ct)$ explicit

(2) $u(x,t) = f(x - u(x,t)t)$ implicit.

(1) $u_t + c u_x = 0$

$u(x,0) = \cos(kx)$, $k = \frac{2\pi}{\lambda}$ spatial freq

$u(x,0) = e^{ikx}$

$e^{i\theta} = \cos\theta + i\sin\theta$, $i = \sqrt{-1}$

ansatz: $u(x,t) = a(t) e^{ikx}$

$\frac{da}{dt} + ikca = 0$

$\frac{da}{dt} = -ikca \Rightarrow a(t) = e^{-i\omega t}$, $\omega = kc$ temp. freq

$a(0) = 1$

$u(x,t) = e^{i(kx - kct)} = e^{ik(x-ct)}$ ↑ traveling Fourier mode

Now the conservation of mass equation can be rewritten as

$$\boxed{\rho_t + \nabla\cdot(\rho\vec{u}) = 0} \Rightarrow \boxed{\frac{1}{\rho}\frac{D\rho}{Dt} = -(\nabla\cdot\vec{u})}$$

if flow locally divergent: $\nabla\cdot\vec{u} > 0$ 

if flow locally convergent: $\nabla\cdot\vec{u} < 0$ 

We said flow is incompressible when

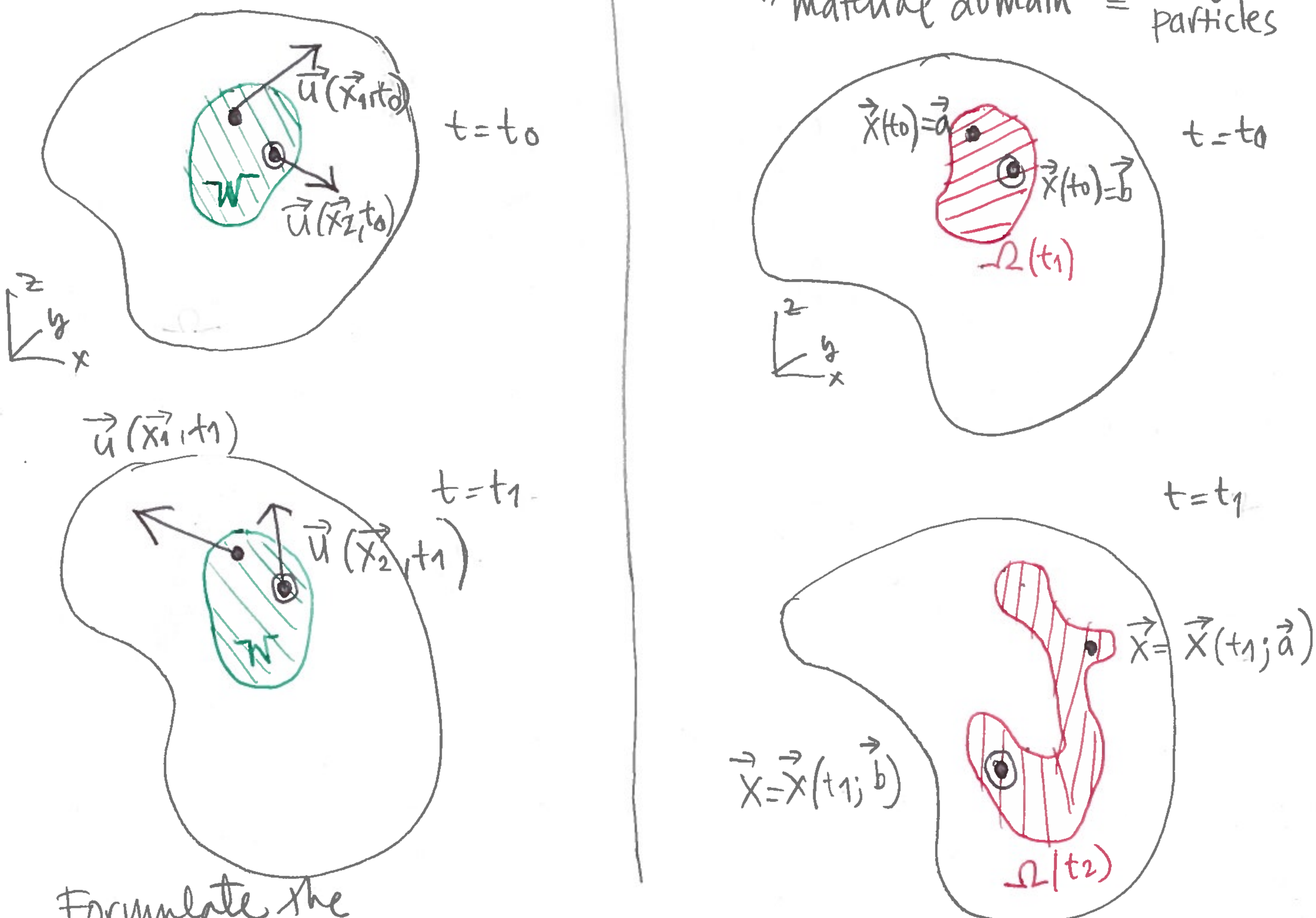
$$\boxed{\nabla\cdot\vec{u} = 0} \text{ in } \Omega \Rightarrow \boxed{\frac{1}{\rho}\frac{D\rho}{Dt} = 0}$$

(a) says $\rho(x,0) = \rho_0$, constant, stays constant

(b) 1D: $\rho_t + u(x(t;a),t)\,\rho_x = 0$

- density value flowing with the particle that started at $x=a$ does not change
- in the moving frame of particle "a" ρ = constant in time.

Eulerian formulation	Lagrangian formulation
fixed control domain	moving control domain
	"material domain" ≡ same particles



Formulate the Conservation of Mass in the Lagrangian frame:

$$\boxtimes_1 \quad \boxed{\frac{d}{dt}\, m(t;\Omega(t)) = \frac{d}{dt}\int_{\Omega(t)} \rho(\vec{x},t)\, d\Omega(t) = 0}$$

no flux term. (compare with page 3)

? to get a PDE having in mind $\Omega=\Omega(t)$ has been chosen arbitrarily.

Define $\boxed{\varphi_t(\Omega(t_0)) = \Omega(t)}$ which maps the initial material domain $\Omega(t_0)$ onto the material domain at a later time t

Namely
$$\vec{a} \xmapsto{\varphi_t} \vec{x}(t;\vec{a}) \quad \text{orbit of particle "}\vec{a}\text{"}$$
$$\vec{b} \mapsto \vec{x}(t;\vec{b}) \quad \text{orbit of particle "}\vec{b}\text{".}$$

etc....

generically $\varphi_t(\vec{x}(t_0)) =$ gives the orbit of this particle that at time $t=t_0$ was at $\vec{x}(t_0)$.

Going back to $\boxtimes_1$, let $\boxed{\Omega_0 = \Omega(t_0);}$

Jacobian

$$\boxed{\frac{d}{dt} m(t;\Omega(t)) = \frac{d}{dt}\int_{\Omega_0} \rho(\varphi_t(\vec{x}(t_0)),t)\, J\, d\Omega_0 =}$$

$$= \int_{\Omega_0}\left[\frac{d\rho}{dt}J + \rho\frac{dJ}{dt}\right] d\Omega_0$$

fixed.

Lemma: $\dfrac{dJ}{dt} = (\nabla\cdot\vec{u})J$.
$J(0) = J_0$

$$J = \frac{\partial(x,y,z)}{\partial(x_0,y_0,z_0)} = \begin{vmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial y}{\partial x_0} & \cdots \\ \frac{\partial x}{\partial y_0} & & \\ \vdots & & \end{vmatrix}$$

Therefore

proper notation

$$\frac{dm}{dt} = \int_{\Omega_0}\left(\frac{d\rho}{dt} + \rho(\nabla\cdot\vec{u})\right)J\,d\Omega_0 = \int_{\Omega(t)}\left(\frac{D\rho}{Dt} + \rho(\nabla\cdot\vec{u})\right)d\Omega(t) = 0$$

$\leftarrow$ for any material domain

$$\Rightarrow \boxed{\frac{D\rho}{Dt} + \rho(\nabla\cdot\vec{u}) = 0} \quad \text{as before.}$$

By a similar procedure (a little more involved) one gets equations for the balance of momentum basically through an integral version of Newton's 2nd law ; mass · acceleration $= \sum$ Forces.

We get Euler's equations :

$$\begin{cases} \rho \dfrac{D\vec{u}}{Dt} = -\nabla p + \rho \vec{b} & \text{(balance of momentum)} \\ \dfrac{1}{\rho}\dfrac{D\rho}{Dt} = -\nabla \cdot \vec{u} & \text{(conservation of mass)} \end{cases}$$

where $\vec{b} = (0,0,-g)$ is the typical mass force due to gravity

Homogeneous fluid $\rho \equiv \rho_0$

Incompressible fluid $\nabla \cdot \vec{u} = 0$

– 2D – 3x3 system — difficult to solve —

$$\begin{cases} u_t + u u_x + v u_y = -\dfrac{1}{\rho_0} p_x \\ v_t + u v_x + v v_y = -\dfrac{1}{\rho_0} p_y - g \\ u_x + v_y = 0 \quad \text{(incompressibility)} \end{cases}$$

2D: Linear + incompressible + irrotational.

Solve first for u and v, and then get pressure at the end,

$$\begin{cases} u_x + v_y = 0 & \text{(incompr.)} \\ v_x - u_y = 0 & \text{(irrotational)} \end{cases}$$

potential flow as we will see ....

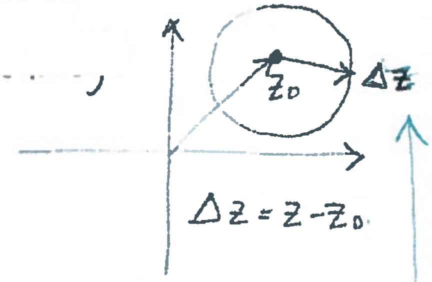
We can now use COMPLEX VARIABLES!

How?

Complex Variables REVIEW

DERIVATIVES:

$$\boxed{f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}}$$



Limit exists $\Rightarrow$ f is differentiable at z_0. If $f'(z)$ exists in a neighborhood of z_0 then $f(z)$ is analytic in that neighborhood.

alternative notation $w = f(z)$

$$\boxed{f'(z) = \frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}}$$

in any direction

- Ex: $f(z) = z^2$

$$\lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z^2 + 2z\Delta z + \Delta z^2) - z^2}{\Delta z} = 2z$$

- Ex: $f(z) = \bar{z}$

$$\lim_{\Delta z \to 0} \frac{\overline{z + \Delta z} - \bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

$\to = 1 \quad \Delta y = 0$

$\to = -1 \quad \Delta x = 0$

$\to = -i \quad \Delta x = \Delta y$

nowhere differentiable!

true for $\forall z$.

DIFFERENTIATION FORMULAS are the same as with real valued functions. Usual rules apply.

$$\frac{d}{dz}\left[f(z) + g(z)\right] = f' + g'$$

$$\frac{d}{dz}\left[f(z)\, g(z)\right] = f'g + fg' \quad \text{(product rule)}$$

$$\frac{d}{dz}\left[g(f(z))\right] = \underbrace{g'(f(z))}_{g'(w)\big|_{w = f(z)}} f'(z) \quad \text{(chain rule)}$$

CAUCHY-RIEMANN equations

$$\frac{df}{dz} = f'(z_0) = \lim_{z \to z_0} \frac{\Delta w}{\Delta z}$$

↑ independent of direction through which z approaches z_0.

Therefore

if $\boxed{\Delta y \equiv 0}$ (identically zero during calculation)

$$\frac{df}{dz} = \frac{\partial}{\partial x}\left(u(x,y) + i\,v(x,y)\right)\Big|_{z=z_0} = \boxed{u_x(x_0,y_0) + i\,v_x(x_0,y_0)}$$

if $\boxed{\Delta x = 0}$

$$\frac{df}{dz} = \lim_{\substack{z \to z_0 \\ \Delta z = \cancel{\Delta x} + i\Delta y}} \frac{\Delta w}{i\Delta y} = -i\frac{\partial}{\partial y}(u + iv)\Big|_{z=z_0} =$$

$$= \boxed{-i\left(u_y(x_0,y_0) + i\,v_y(x_0,y_0)\right)}.$$

By equating we get a system of Partial Diff. Eqns (PDE) known as the $\boxed{\text{Cauchy-Riemann eqns.}}$

short notation
$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

$$\begin{cases} \dfrac{\partial u}{\partial x}(x,y) = \dfrac{\partial v}{\partial y}(x,y) \\[2ex] \dfrac{\partial u}{\partial y}(x,y) = -\dfrac{\partial v}{\partial x}(x,y) \end{cases}$$

longer notation.

Complex Variables X PDEs

- Important fact:

$\boxed{f(z) \text{ analytic in } D \quad (\forall z \in D) \quad \text{Then } u \text{ and } v \text{ are harmonic in } D.}$

for all — domain D; open set — in

Harmonic functions satisfy Laplace's eqn. (A PDE $\Delta u = 0$)
$\Delta \equiv \partial_{xx} + \partial_{yy}$

Application in (steady) temperature distributions
in Fluid Dynamics
in Electrostatic potential
etc --- to be seen

- Check : Cauchy-R. $\begin{cases} u_x = v_y \\ u_y = -v_x. \end{cases}$

$$\oplus \boxed{\begin{bmatrix} u_{xx} = v_{yx} \\ u_{yy} = -v_{xy} \end{bmatrix}} \quad \begin{matrix} u_{xy} = v_{yy} \\ u_{yx} = -v_{xx} \end{matrix} \quad \ominus . \quad \boxed{\begin{cases} \Delta u = 0 \\ \Delta v = 0 \end{cases}}$$

- Note: Continuity of 2nd. derivatives imply that order can be interchanged, namely
$u_{xy} = u_{yx}$ etc...

• Easy because $f(z)$ analytic $\Rightarrow$ infinitely smooth in region of analyticity (to be verified).

Let u, v be harmonic in D.
Let u, v satisfy the Cauchy-Riemann eqs. $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

then v is said to be the HARMONIC CONJUGATE of u.

"meaning it is the imaginary part of an analytic func. that has u as its real part..

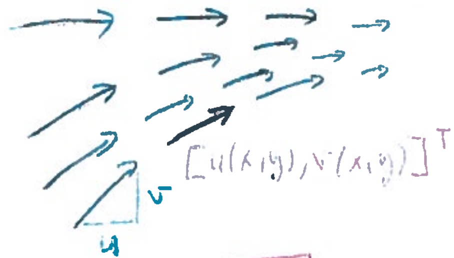
— FLUIDS —

Lets consider HYDRODYNAMICS, namely water. In many cases we can consider the flow to be:

$\boxed{\text{(A) • incompressible}}$

$\boxed{\text{(B) • irrotational}}$

$\vec{u} = [u, v]^T$ is the velocity field in the fluid



(A) This can be expressed with the $\boxed{\text{divergence-zero}}$ condition

$$\text{div}\,\vec{u} = \boxed{\nabla\cdot\vec{u}} = [\partial/\partial x, \partial/\partial y]\,[u, v]^T = \boxed{u_x + v_y = 0}$$

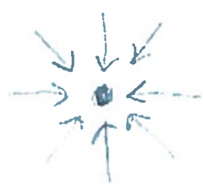
Recall from vector calculus:

(A1) $\nabla\cdot\vec{u}(x_0, y_0) > 0$ locally field looks like

$\bullet \equiv (x_0, y_0)$

divergent field

(A2) $\nabla\cdot\vec{u}(x_0, y_0) < 0$



convergent field

in fluids $\equiv$ a source (A1) or a sink (A2) of fluid.

in electrostatics — where $\vec{u}$ expresses the electric field

then we have a positive (A1) charge density and a negative (A2) charge density

Clearly DIVERGENCE of a vector field expresses the LOCAL RATE of its divergence or convergence.

Recall Divergence Theorem (Gauss Theorem).

$$\boxed{\iint_D (\nabla\cdot\vec{u})\,dA = \int_C \underbrace{\vec{u}\cdot\vec{n}\,ds}_{\text{normal flux through } C},}$$

(B) irrotational means curl-free:

$$\boxed{\text{curl}\,\vec{u} = \nabla \times \vec{u} = 0}$$

also called the VORTICITY of the flow

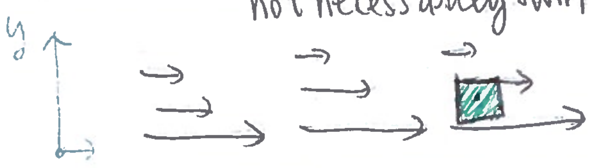
curl = expresses the local rate of rotation of a vector field (see Stokes Theorem)

For 2D vector fields

$$\text{curl}\,\vec{u} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & 0 \\ u & v & 0 \end{vmatrix} = \boxed{(v_x - u_y)\,\vec{k}}$$

can be treated as a scalar

• Example: has rotation
not necessarily swirling



will rotate square parcel of fluid.
(bottom part moves faster)

Here $\boxed{u = u_0 - ay \;,\; v = 0}$ $\boxed{\text{shearing flow}}$

$\text{curl}\,\vec{u} = 0 - (-a) = a$

(TOP VIEW) like this everywhere.

in summary: (A) $u_x + v_y = 0$ or $\boxed{u_x = -v_y}$
(B) $v_x - u_y = 0$ or $\boxed{u_y = v_x}$

This looks almost as Cauchy-Riemann equations

$$\boxed{\begin{aligned} u_x &= (-v)_y \\ u_y &= -(-v)_x \end{aligned}}$$

(There are different ways to do this ...)

As a short cut let me propose the following complex function, called the complex velocity potential:

$$\boxed{\Phi(z) = \phi(x,y) + i\psi(x,y)}$$

where ϕ is our usual velocity potential with

$$[u,v] = \nabla\phi$$

and ψ its $\boxed{\text{harmonic conjugate}}$, is called the STREAM FUNCTION for reasons that we will see soon.

When $\Phi(z)$ is ANALYTIC, what do we get?

CR

$\phi_x = \psi_y$

$\phi_y = -\psi_x$

$$\frac{d\Phi}{dz} = \overbrace{\phi_x + i\psi_x}^{\partial x} = \overbrace{-i(\phi_y + i\psi_y)}^{-i\,\partial y} = \psi_y - i\phi_y$$

$$\frac{d\Phi}{dz} = u + i\psi_x = \psi_y - iv$$

We learn from Cauchy-Riemann equations that

(1) $$\frac{d\Phi}{dz} = u - iv = \overline{\left(\frac{dz}{dt}\right)} \equiv \text{the complex velocity}$$

(−1)·vertical ($v = \phi_y$)

horizontal ($u = \phi_x$)

(2) $[u,v] = [\psi_y, -\psi_x] \equiv \nabla^\perp\psi$, $\nabla^\perp = (\partial y, -\partial x)$ = "grad perp"

(3) level curve of the STREAMFUNCTION ψ is

$\psi(x,y) = \text{constant}$

parametrize it

$$\psi(x(s), y(s)) - \text{Constant} = 0$$

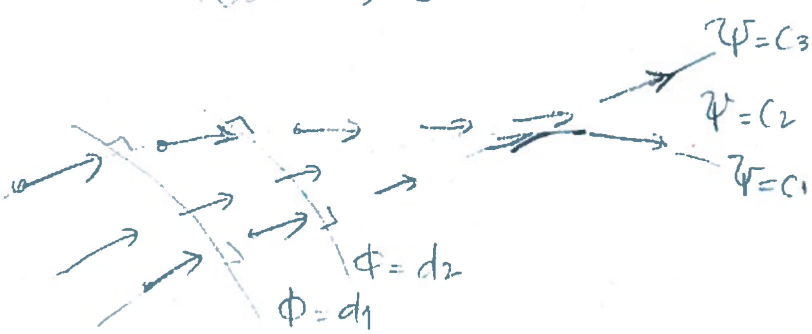
then

$$\frac{d\psi}{ds} = \boxed{\psi_x \frac{dx}{ds} + \psi_y \frac{dy}{ds} = 0}$$

$$-v\frac{dx}{ds} + u\frac{dy}{ds} = 0 \implies \boxed{[u,v]\underbrace{\left[\frac{dy}{ds}, -\frac{dx}{ds}\right]^T}_{\vec{n} \text{ to } \psi = \text{Const.}} = 0}$$

Conclusion the level curve ...

... $\boxed{\psi(x,y) = c}$ is parallel to velocity field, and helps us visualize the streaming flow $\implies$ STREAMFUNCTION name for ψ.



(4) $\nabla\phi$ = normal to $\phi(x,y) = \text{Constant}$
$\nabla\psi$ = normal to $\psi(x,y) = \text{Constant}$

$$\underbrace{\begin{matrix}\nabla\phi = [u,v] \\ \nabla\psi = [-v,u]\end{matrix}}$$

$$\boxed{\nabla\phi \perp \nabla\psi}$$

(5) Having all this in place.

$\boxed{\Phi(z) = \text{analytic}}$ automatically takes into account that the flow is **irrotational** and **incompressible**:

$$\oint_G \Phi'(z)\,dz = \underset{\substack{\uparrow \\ \text{no circulation} \\ \text{(swirling)}}}{0} + i \underset{\substack{\uparrow \\ \text{no mass flux} \\ \text{across boundary}}}{0}$$

G = any closed simple curve

(6) From (1) $\dfrac{d\Phi}{dz} = u - iv$ analytic

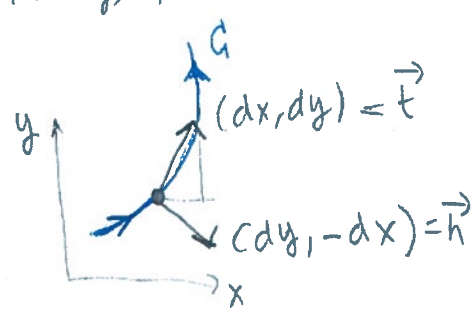
$\Rightarrow$ Cauchy-Riemann equations

$$\begin{cases} u_x = (-v)_y & \Longrightarrow \quad u_x + v_y = 0 \quad (\text{incompr.}) \\ u_y = -(-v)_x & \Longrightarrow \quad v_x - u_y = 0 \quad (\text{irrotat.}) \end{cases}$$

- Take an arbitrary contour G:

$$\boxed{\oint_G \Phi'(z)\,dz} = \oint_G (u - iv)(dx + i\,dy) = \underbrace{\oint_G (u\,dx + v\,dy)}_{\text{circulation along contour } G} +$$

$$+ i \underbrace{\oint_G (u\,dy - v\,dx)}_{\text{flux through contour}} = \boxed{0}$$



$\dfrac{d\Phi}{dz}$ is analytic + Cauchy-Goursat theorem

$$\oint_G (\text{analytic function})\,dz = 0$$