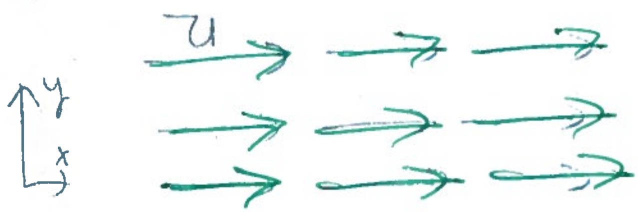
If we want to solve a problem in a complicated domain such as D_z and we know the mapping to a simpler domain D_w, where we can find the solution more easily, then a composition such as above does the job.

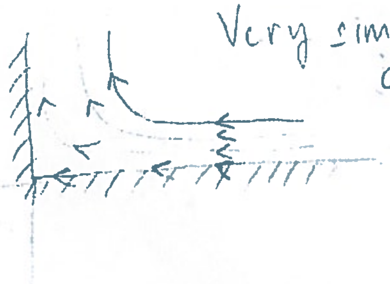
Some Fluid Flow problems with Conformal Mapping:

Ex 1 What is the complex velocity potencial for the uniform horizontal flow with speed $U>0$?



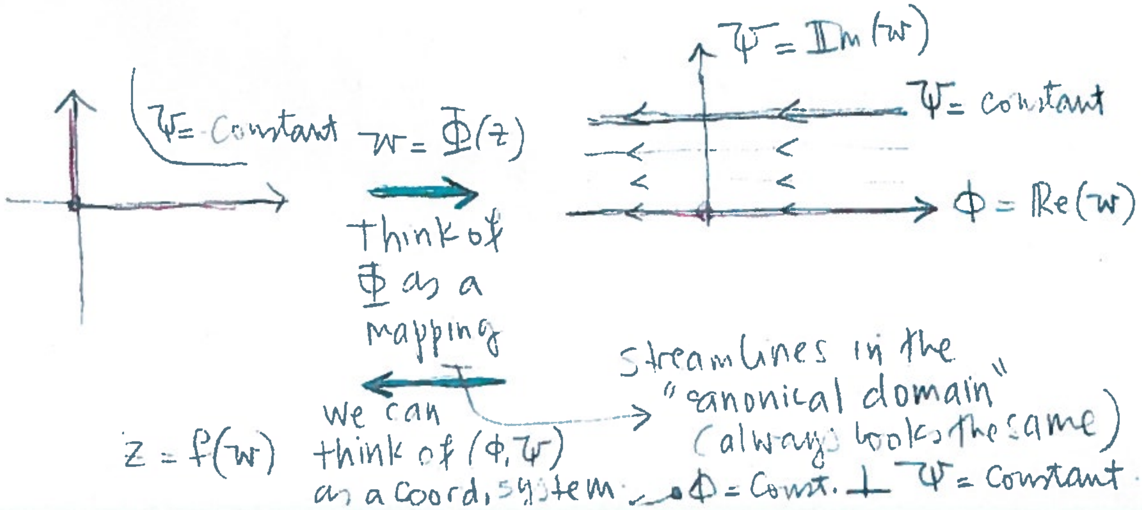
$$\frac{d\Phi}{dz} = u - iv = U \Rightarrow \Phi(z) = Uz$$

Ex 2: Very simplified model of a turning flow at a corner

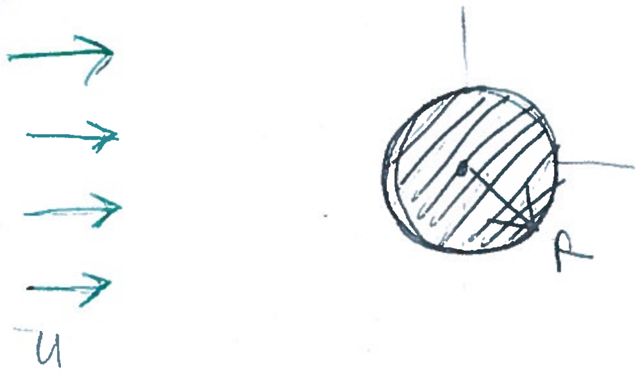


Find a complex velocity potential $\Phi(z)$.

$W = \Phi(z) = \phi + i\psi$



Ex 3. How about the flow around a cylinder?



Milne-Thomson Circle Theorem:

Let the free velocity potential be given by $\boxed{f(z)}$, analytic in the region $|z| < R$. Then in the presence of a cylinder of radius R, centered at $z=0$, the complex velocity potential is given by

$$\boxed{\Phi(z) = f(z) + \overline{f\left(\frac{R^2}{\bar{z}}\right)}}$$

Facts

(A) Along the circle $(z = Re^{i\theta})$ we have that

$$\boxed{\Phi(Re^{i\theta})} = f(Re^{i\theta}) + \overline{f\left(\frac{R^2}{Re^{-i\theta}}\right)} =$$

$$= f(Re^{i\theta}) + \overline{f(Re^{i\theta})} = \boxed{f(z) + \overline{f(z)}}$$

$$= \boxed{2\,\mathrm{Re}\, f(z) \implies \mathrm{Im}\,\Phi(z) = 0}$$

$$\boxed{\psi(x,y) = 0 = \text{constant}}$$

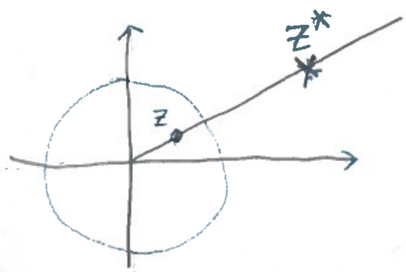
Boundary of cylinder is a streamline!

the velocity field is tangent to streamlines (level curves $\Psi(x,y) = \text{constant}$)

$\mapsto$ therefore we conclude from the Circle theorem that the fluid goes around the cylinder.

(B) We saw that if $f(z)$ is analytic then $\overline{f(\bar{z})}$ is analytic in the same domain D, namely for $z \in D$

(C) The point $\boxed{z^* = \dfrac{R^2}{\bar{z}}}$ is called the symmetric point (or image point) of z with respect to the circle.



as $z \to 0$, $z^* \to \infty$

image of $\boxed{z=0}$ is at infinity

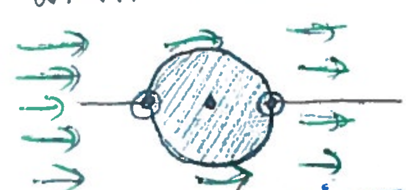
image of $z = Re^{i\theta}$ "coincides with itself"

Uniform flow case + a cylinder (use Circle Theorem)

(d) $\boxed{f(z) = Uz}$ $\quad \overline{f\left(\frac{R^2}{\bar{z}}\right)} = U\left(\overline{\frac{R^2}{\bar{z}}}\right) = U\frac{R^2}{z}$

$$\boxed{\Phi(z) = U\left(z + \frac{R^2}{z}\right)}$$

$$\boxed{\frac{d\Phi}{dz}(z) = U\left(1 - \frac{R^2}{z^2}\right)}$$, $u - iv = 0$ when $z = \pm R$ stagnation points



STAGNATION POINTS at $z = \pm R$

$$\boxed{\int_{C_R} \frac{d\Phi}{dz}(z)\, dz = 0}$$ $\Rightarrow$ no circulation, no mass flux

not analytic

$$\Phi(z) = U\left(z + \frac{R^2}{z}\right) = Uz + UR^2\frac{\bar{z}}{|z|^2}$$

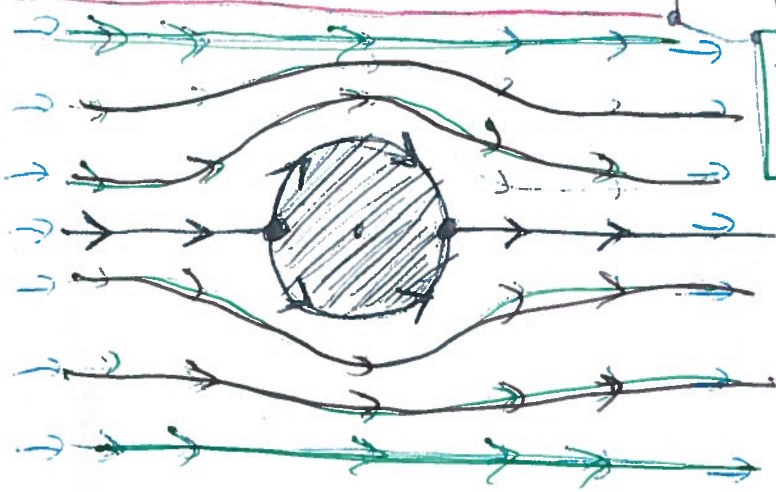
$$= \left(Ux + \frac{UR^2x}{x^2+y^2}\right) + i\left(Uy - \frac{UR^2y}{x^2+y^2}\right)$$

$$\boxed{\psi(x,y) = Uy\left(1 - \frac{R^2}{x^2+y^2}\right)}$$

$\psi = 0$, $y = 0$

$\psi = 0$, $x^2+y^2 = R^2$

$$\boxed{\psi = U\left(r - \frac{R^2}{r}\right)\sin\theta}$$



Recall $u = \psi_y$ (cx 4a)

- when $|y|$ is large

$$\psi(x,y) = Uy\left(1 - \frac{R^2}{x^2+y^2}\right) \approx Uy(1-0)$$

streamline $\approx$ horizontal → see green streamline

- when $|x|$ is large

$$\psi(x,y) = Uy\left(1 - \frac{R^2}{x^2+y^2}\right) \approx Uy(1-0)$$

and flow is approximately uniform $\Rightarrow$ U

- Complex velocity potential for a point vortex

(we saw) 2D-Laplace fund. soln.; arctan y/x (harmonic conjugate); check $\Phi = \log z$ source.

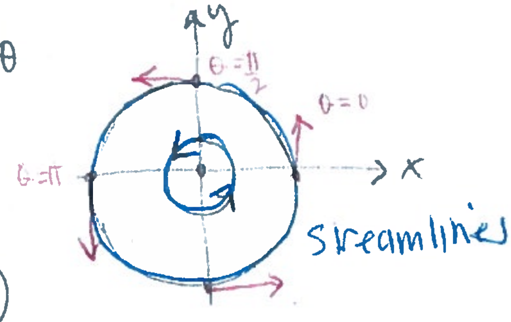
M1 $\boxed{\Phi(z) = -i\underset{\text{strength}}{a}\log(z)} = -ia(\ln r + i\arg z)$

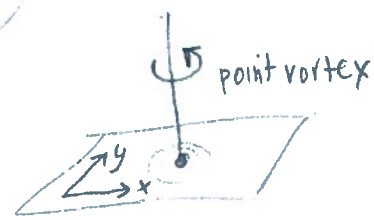
$$\Phi(z) = a\arg z - ia\ln r = \phi(x,y) + i\psi(x,y)$$

$$\frac{d\Phi}{dz} = -i\frac{a}{z} = -i\frac{a}{re^{i\theta}} = -i\frac{a}{r}e^{-i\theta}$$

$$\frac{d\Phi}{dz} = u - iv = -\frac{a}{r}(\sin\theta + i\cos\theta)$$

$$\boxed{u = -\frac{a}{r}\sin\theta, \quad v = \frac{a}{r}\cos\theta}$$

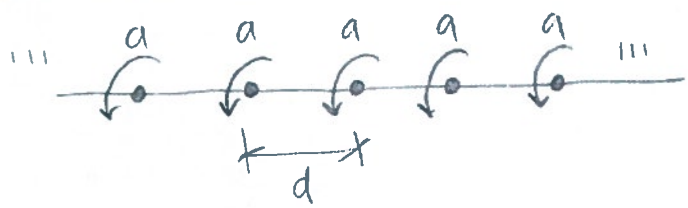




- velocity singular at the origin

macroscopic ("far away") model for a cylindrical eddy which has collapsed to a line.

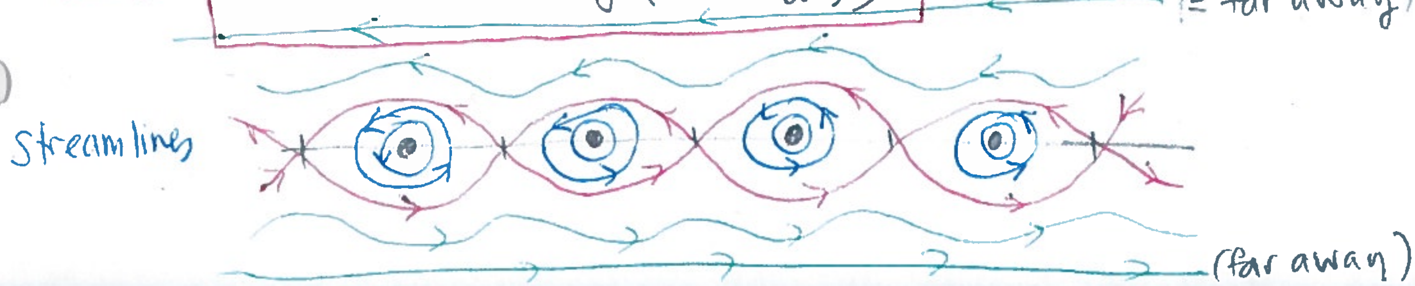
- Now take

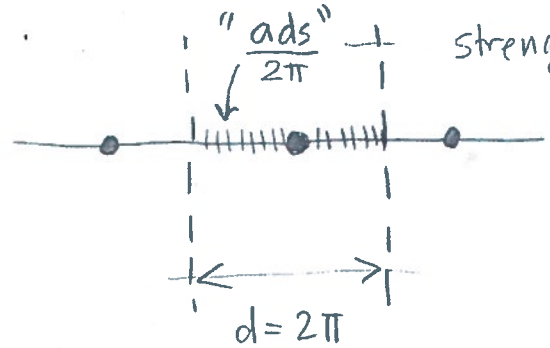


an infinite line of point vortices with a periodic structure

It can be shown that

M2 $$\boxed{\Phi(z) = -ia\log\left(\sin\left(\frac{\pi z}{d}\right)\right)}$$





strength of the infinitesimal vorticity

let

"point vortex" center at s.

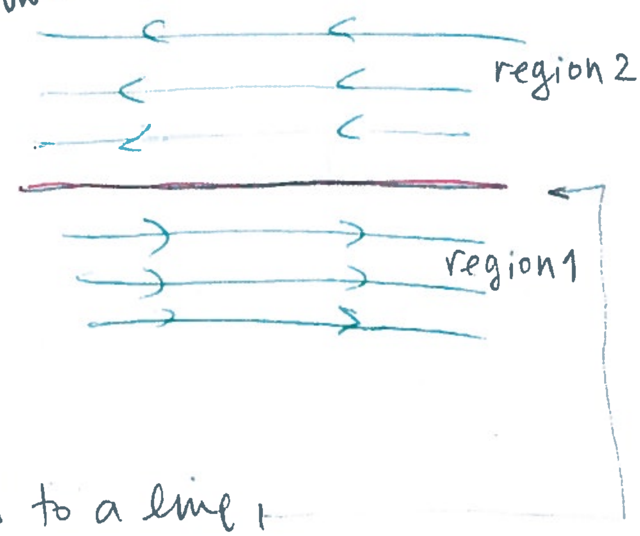
M3

$$\boxed{\Phi(z) = -\frac{i}{2\pi}\int_0^{2\pi} a \log\left(\sin\left(\frac{z-s}{2}\right)\right) ds}$$

$d = 2\pi$

"adding" all point vortices within periodic window.

This models the flow, ⟶



namely a shear flow.

Compare with figure for M2. This is like a macroscopic model for M2 where the recirculation cells collapsed to a line.

This line is a VORTEX SHEET.

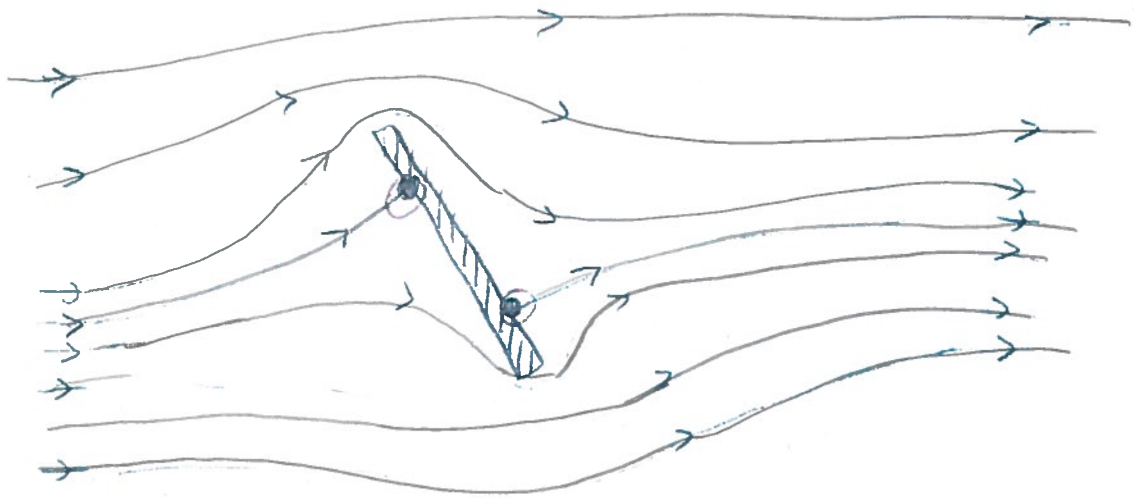
$\nabla \times \vec{u} = \nabla \times (u, v) = (v_x - u_y)\vec{k} = \omega \vec{k}$; where $v \equiv 0$

region 2 : $u \equiv$ constant $\Rightarrow$ $\omega =$ vorticity $= 0$
region 1 : $u \equiv$ constant $\Rightarrow$ $\omega = 0$
along the interface : $u(x,y) = -(u_2 + u_1)\,H(y)$, Heaviside.

$$\boxed{\omega = -u_y = (u_2 + u_1)\,\delta(y)}$$

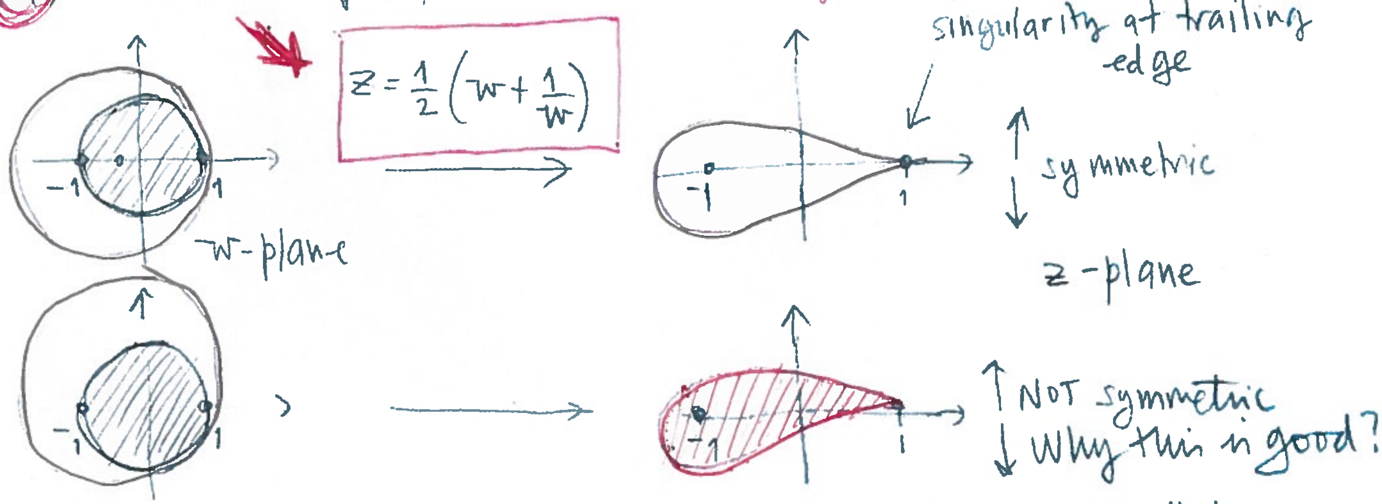
singular distribution of vorticity.

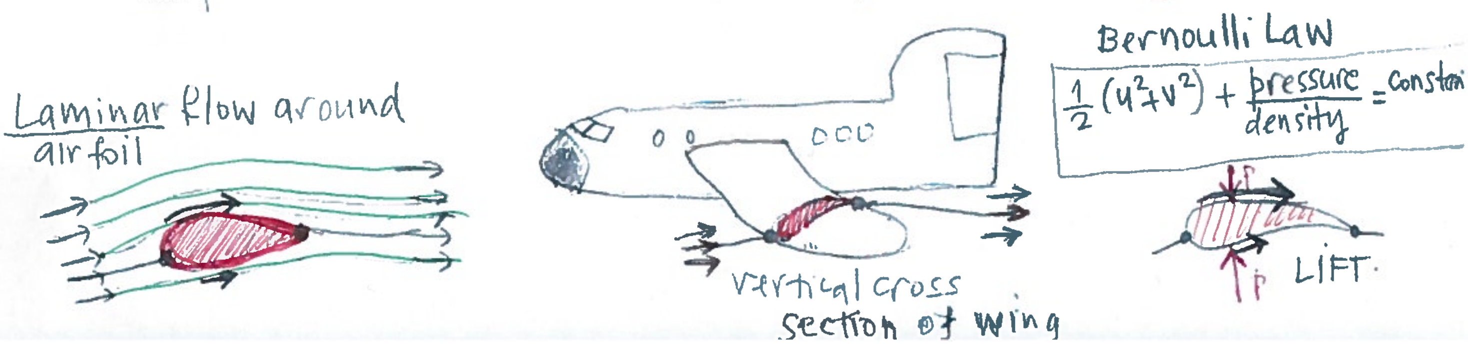
(c) putting all together for the finite length plate, including circulation and the angle of attack we can express through complex variables a flow like



- Stagnation points, one in front, one in the back.

(d) other examples / facts: Joukowsky transformation



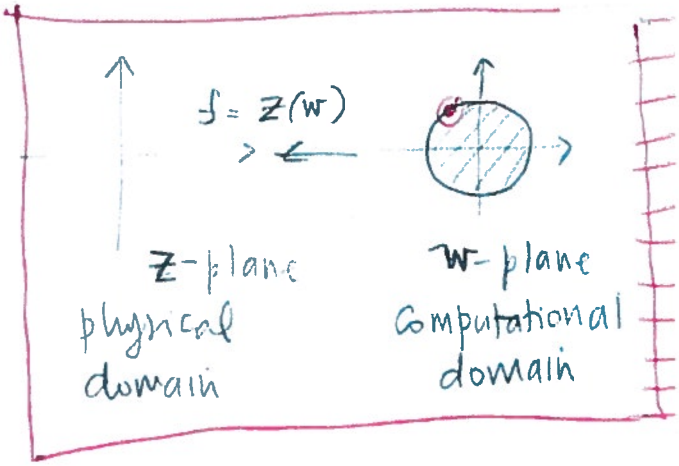


Consider the mapping

$$\boxed{z = w + \frac{\lambda^2}{w}}$$

$\lambda \in \mathbb{R}$, constant

Joukowsky transf.



let $\boxed{w = Re^{i\theta}}$

$$z = Re^{i\theta} + \frac{\lambda^2}{Re^{i\theta}} = \left(R + \frac{\lambda^2}{R}\right)\cos\theta + i\left(R - \frac{\lambda^2}{R}\right)\sin\theta$$

call this x

call this y

and note that

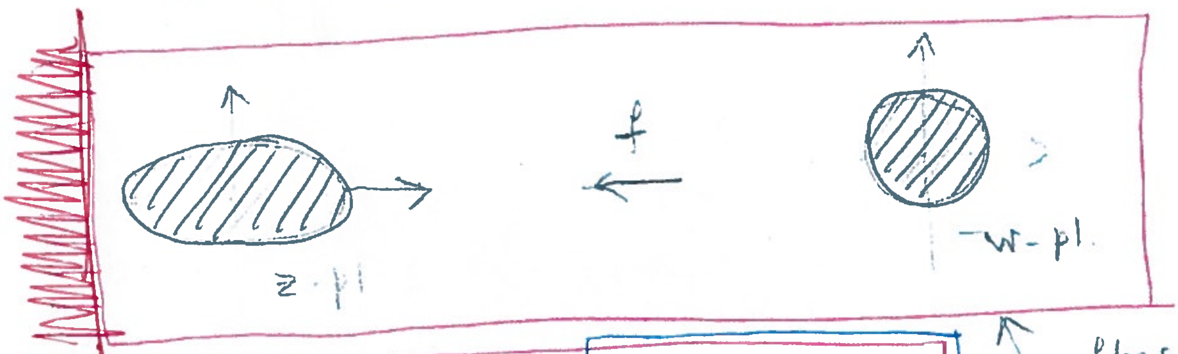
$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

where

$$a = R + \frac{\lambda^2}{R}$$

$$b = R - \frac{\lambda^2}{R}$$

ellipse with axis equal to 2a and 2b.



$$\Phi(w) = u\left(w + \frac{R^2}{w}\right) + \boxed{i\frac{\kappa}{2\pi}\log\left(\frac{w}{R}\right)}$$

add circulation.

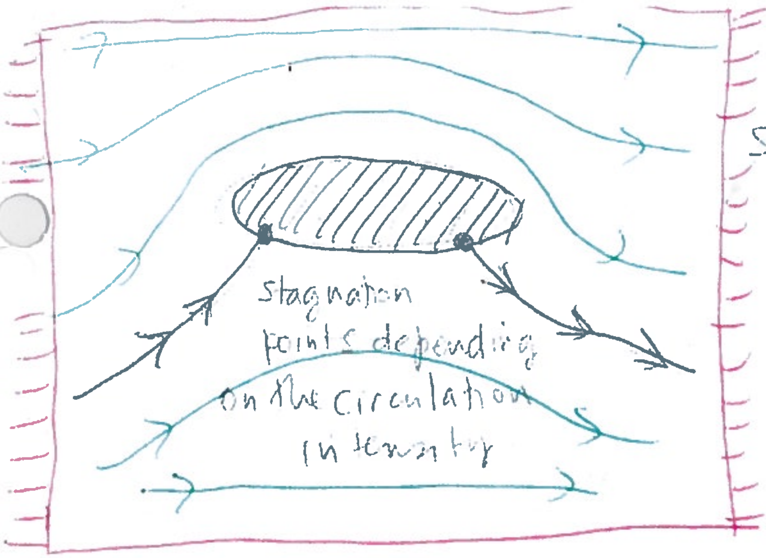
flow around cylinder of circular cross section (w-plane)

while

$$\boxed{\Phi(z) = u\left(w(z) + \frac{R^2}{w(z)}\right) + i\frac{\kappa}{2\pi}\log\left(\frac{w(z)}{R}\right)}$$

in the z-plane

is the complex velocity potential for the cylinder with elliptical cross section



Still we need to write $w = w(z)$

$$z(w) = w + \frac{\lambda^2}{w}$$

$$zw = w^2 + \lambda^2$$

$$-w^2 - zw + \lambda^2 = 0 \qquad \text{complex coeff}$$

2 roots

$$w = \frac{z + \sqrt{z^2 - 4\lambda^2}}{2}$$

$$\boxed{w(z) = \frac{z}{2} + \frac{1}{2}\left(z^2 - 4\lambda^2\right)^{1/2}}$$

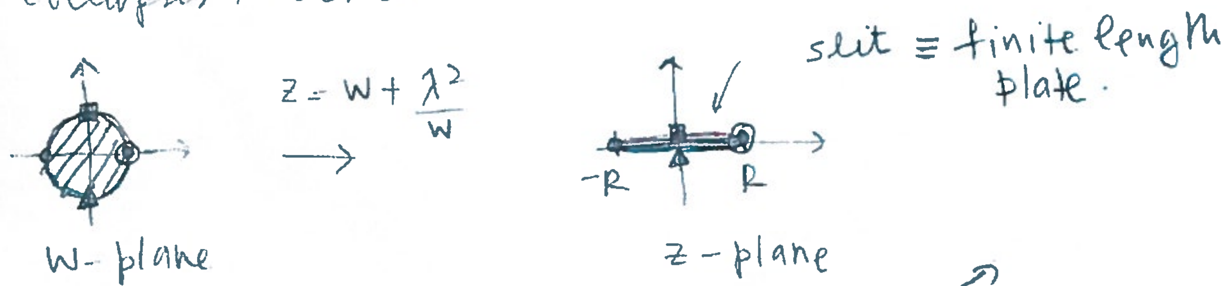
The mapping

$$\boxed{z = w + \frac{\lambda^2}{w}}$$

is called the Joukowsky transformation, an important name in classical airfoil theory in Aerodynamics (Nikolai E. Zhukovsky 1847-1921)

◉ Special cases

(A) When $\lambda^2 = R$ note that one of the ellipse's axis collapses to ZERO.

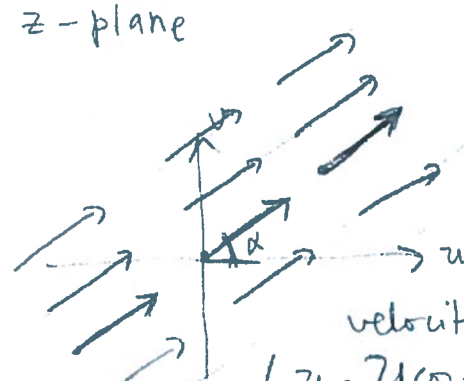


(B) $\Phi(z) = \mathcal{U} e^{-i\alpha} z$

$$\frac{d\Phi}{dz} = \mathcal{U}\cos\alpha - i\,\mathcal{U}\sin\alpha$$

$$\left|\frac{d\Phi}{dz}\right| = \mathcal{U}$$

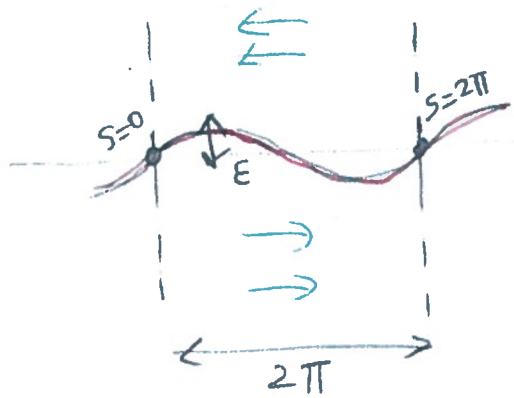
α = angle of attack



$$\begin{cases} u = \mathcal{U}\cos\alpha \\ v = \mathcal{U}\sin\alpha \end{cases}$$

Now a $\boxed{\text{hard question}}$: is this interface STABLE to perturbations?

Lets consider a simple configuration: small periodic perturbations, namely by a single Fourier mode.



parametrization along interface.

This problem relates to underlying 2D incompressible Euler eq. with non-smooth data. → investigated with this compact set of diff. eqs.

An integral-differential evolution system:

$$\begin{cases} \dfrac{\partial \bar{z}}{\partial t}(s,t) = \dfrac{1}{4\pi i}\displaystyle\int_0^{2\pi} \dfrac{\partial \Gamma}{\partial s}(s',t)\cot\left[\dfrac{z(s,t)-z(s',t)}{2}\right] ds' \\[2ex] \dfrac{\partial \Gamma}{\partial t}(s,t) = \dfrac{\sigma}{\rho} K(s,t), \end{cases}$$

$$K(s,t) = \frac{x_s y_{ss} - x_{ss} y_s}{(x_s^2 + y_s^2)^{3/2}} \equiv \text{curvature}$$

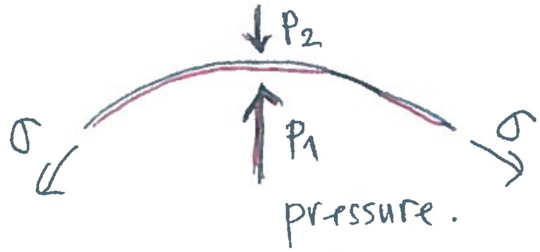
σ = surface tension
ρ = density

Remarks:

1. $\dfrac{\partial \Gamma}{\partial s}(s,t)$ = vortex sheet strength that can change in time and in space (for example NONUNIFORM vortex distribution).
   $\boxed{\Gamma = 0} \Rightarrow$ For example $\boxed{\dfrac{\partial \Gamma}{\partial s}(s,t) \equiv a}$

2. $\dfrac{d\Phi}{dz} = \bar{z} = \underbrace{\dfrac{dx}{dt} - i\dfrac{dy}{dt}}_{\text{velocity of points ON the interface.}} = u - iv$

(3) Second equation comes from Laplace-Young Law:

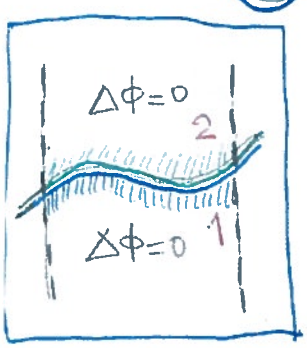


balance of stresses – curvature plays a role.

(4) Last but most importantly, we have a $\boxed{\text{SINGULAR INTEGRAL}}$, and its $\boxed{\text{CAUCHY PRINCIPAL VALUE}}$. Let's learn how to deal with this useful object.

(4a) Where did it come from? It came from differentiating $\boxed{M3}$ with respect to z and letting z approach the interface, in the flat case, approach the real axis.

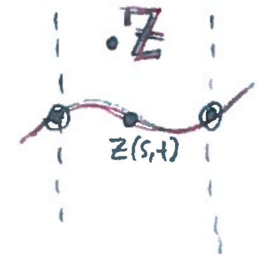
(4b) Why useful? When z (in $\boxed{M3}$) is away from the real axis we have an $\boxed{\text{analytic function}}$ $\Phi(z)$. Therefore the flow is automatically incompressible and irrotational. We automatically satisfy $\boxed{\text{Laplace's equation}}$ for $\phi = Re(\Phi)$.



Another way to see this is that we transformed a $\boxed{\text{2D-problem into a 1D-problem}}$. The flow inside DOMAIN 1 and DOMAIN 2 are easily computed at any time.

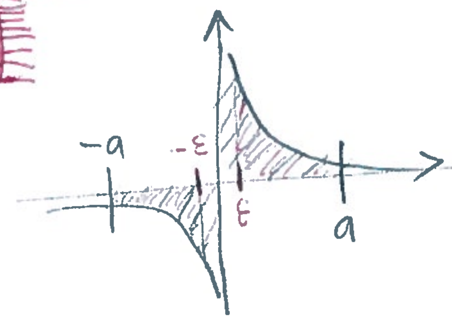
Say use $\boxed{M3}$ in the form

$$\Phi(\mathbf{Z}) = \frac{-i}{2\pi}\int_0^{2\pi} \frac{\partial\Gamma(s,t)}{\partial s}\cot\left[\frac{\mathbf{Z} - z(s,t)}{2}\right] ds$$



(4c) SINGULAR INTEGRAL

$$\int_{-a}^{a} \frac{1}{x}\,dx = ?$$



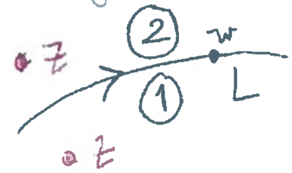
Cauchy Principal Value

$$PV\int_{-a}^{a}\frac{1}{x}\,dx \equiv ⨍_{-a}^{a}\frac{1}{x}\,dx = \lim_{\varepsilon\downarrow 0}\left(\int_{-a}^{-\varepsilon} + \int_{\varepsilon}^{a}\right)\frac{1}{x}\,dx = 0$$ in this case.

(4d) Consider the Cauchy-type integral

Ablowitz & Fokas
Complex Variables
page 517, CUP 2003

$$\boxed{F(z) = \frac{1}{2\pi i}\int_L \frac{\varphi(w)}{w-z}\,dw}$$



where L is a smooth curve (arc or closed contour).
Let the boundary data $\varphi(w)$ be Hölder continuous:

$$\boxed{|\varphi(w_1)-\varphi(w_2)| \le G\,|w_1-w_2|^{\alpha}, \quad 0<\alpha\le 1, \quad G>0}$$

(Note $\alpha = 1 \Rightarrow$ Lipschitz cont.)

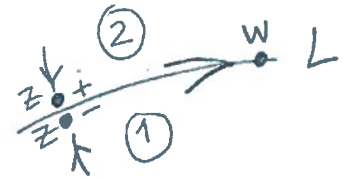
Note $F(z)$ is analytic provided $z \notin L$.

(in Salsa there is a real-valued ($\mathbb{R}^n$) version – page 145)

- **PLEMELJ formula** (Sokhotski – Plemelj theorem)
(1868) (1908)

for the limiting value, of $F(z)$ as z approaches L:

$$\boxed{F^{2,1}(z) = \pm \frac{1}{2}\varphi(z) + \frac{1}{2\pi i}\int_L \frac{\varphi(w)}{w-z}\,dw}$$



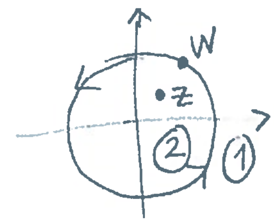
– Therefore note that for $z \in L$ the value of is ambiguous.

– The function $F(z)$ is said to be sectionally analytic:

– Analytic in domain ②
– Analytic in domain ①
– Jumps accross curve separating domains.

– Very convenient object for our modeling!

Example (application): what is $\frac{1}{2\pi i}\int_G \frac{1}{w-z}\,dw$?



We know easily that when

$z \in$ ② $\quad \frac{1}{2\pi i}\int_G \frac{1}{w-z}\,dw = 1$ (winding number of G)

$z \in$ ① $\quad \frac{1}{2\pi i}\int_G \frac{1}{w-z}\,dw = 0$

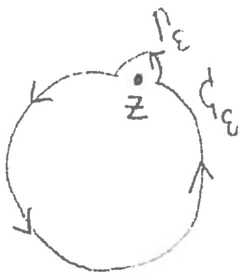
Plemelj: $z \to G$

② $\frac{1}{2} + \frac{1}{2\pi i}\int_G \frac{1}{w-z}\,dw = 1$

① $-\frac{1}{2} + \frac{1}{2\pi i}\int_G \frac{1}{w-z}\,dw = 0$

$\Big\}\ \frac{1}{2\pi i}\int_G \frac{1}{w-z}\,dw = \frac{1}{2}$

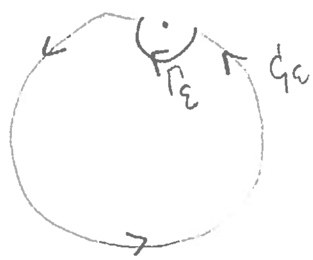
$$\frac{1}{2\pi i}\int_{C} \frac{1}{w-z}\,dw = \text{Cauchy Theorem}$$



"1/2 residue"

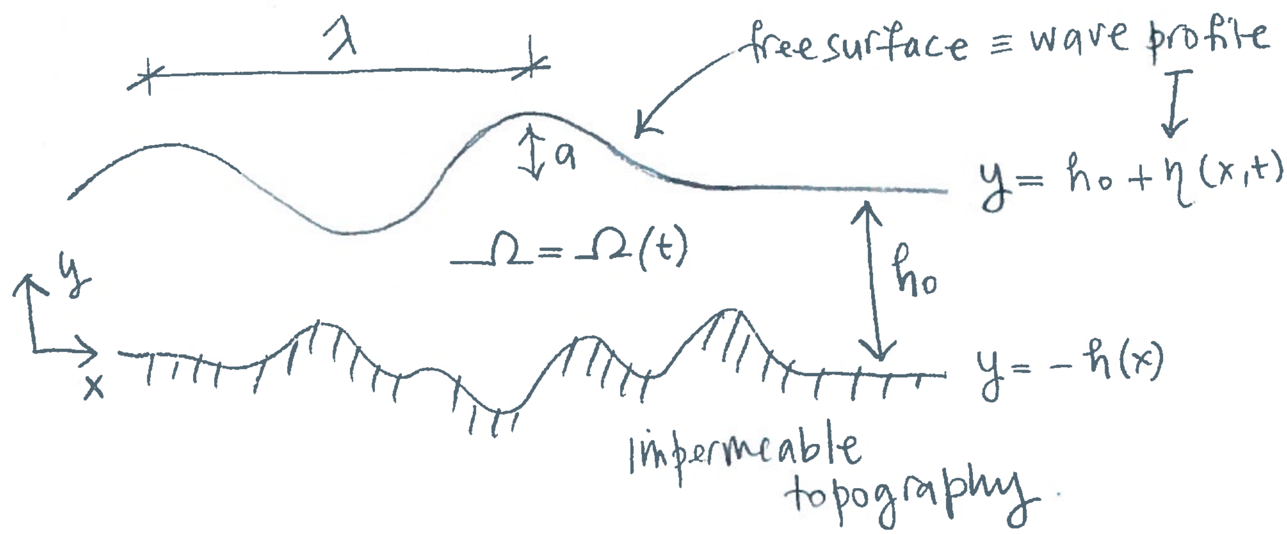
$$\frac{1}{2\pi i}\int_{C} \frac{1}{w-z}\,dw = \frac{1}{2\pi i}\int_{C'_\varepsilon} \frac{1}{w-z}\,dw + \frac{1}{2} = \varepsilon \downarrow 0 = \cdots$$

$$\cdots = \underbrace{\frac{1}{2\pi i} \fint_{C} \frac{1}{w-z}\,dz}_{=1/2} + \frac{1}{2} = 1 \text{ (winding \#)}$$



$$\frac{1}{2\pi i}\int_{C} \frac{1}{w-z}\,dw = \underbrace{\frac{1}{2\pi i} \fint_{C} \frac{1}{w-z}\,dz}_{=1/2} - \frac{1}{2} = 0 \text{ (winding \#)}$$

Water Waves in an incompressible, irrotational regime



Physical / Modeling point of view:

- viscous effects are ignored: no bottom friction.
  no vorticity generation.

- fixed impermeable bottom:
  - no sediment transport
  - bottom porosity ignored — no seepage effect

- Problem of interest from BOTH physical (applications) and mathematical points of view..

Mathematical modeling

2D – nonlinear potential theory

– a free boundary problem

use a velocity potential $\phi(x,y,t)$ where

$$(u,v) = \text{velocity} = \nabla\phi$$

– irrotational : $v_x - u_y = \phi_{yx} - \phi_{xy} = 0$

– incompressible : $u_x + v_y = 0 \Rightarrow \boxed{\Delta\phi = 0}$

– impermeable bottom : no flow

$$(u,v)\cdot\vec{n} = 0$$

$$\nabla\phi\cdot\vec{n} = \boxed{\frac{d\phi}{dn} = 0}$$ Neumann cond.

(Free boundary)

Free surface conditions

Dirichlet data

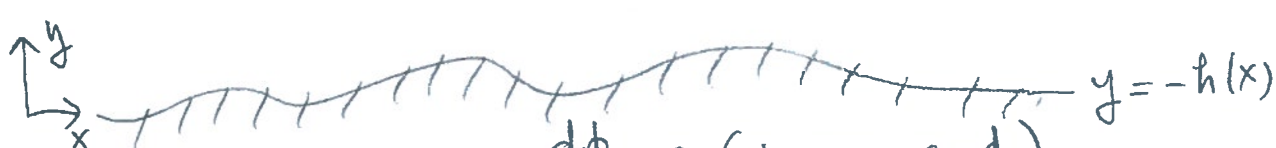
evolution thru the boundary (NONLINEAR)

$$\begin{cases} \phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) + g\eta = 0, & \phi(x, h_0+\eta_0(x), 0) = \phi_0(x) \text{ (Bernoulli Law)} \\ \eta_t + \phi_x\eta_x - \phi_y = 0 & , \eta(x,0) = \eta_0(x) \text{ (Kinematic Condition)} \end{cases}$$

$y = h_0 + \eta(x,t)$

unknown

a nonlocal component $\rightarrow$ $\boxed{\Delta\phi = 0,}$ in $\Omega = \Omega(t)$



$\frac{d\phi}{dn} = 0$ (Neumann Cond.)

– Many interesting problems in PDEs & Applications arise from the system above. Many important PDEs can be obtained from this system: the wave equation, the KdV eq., Boussinesq system, nonlinear Schrodinger eq. among others!

HOW? Start by taking DIMENSIONLESS VARIABLES ($\sim$)

$$\boxed{x = \lambda \tilde{x}, \quad y = h_0 \tilde{y}, \quad t = \frac{\lambda}{c_0}\tilde{t}, \quad \boxed{\eta = a\tilde{\eta}, \quad \phi = \frac{g\lambda a}{c_0}\tilde{\phi}}, \quad h = h_0 \tilde{h}\left(\frac{\lambda}{l}\tilde{x}\right)}$$

where $\lambda \equiv$ (reference/typical) wavelength
$h_0 \equiv$ depth
$c_0 \equiv$ propagation speed
$a \equiv$ amplitude
$l =$ bottom variations' length scale

$$\frac{\lambda}{c_0} = \frac{[L]}{[L]/[T]} = [T] \equiv \text{reference time}$$

reference potential

$$\frac{g\lambda a}{c_0} = \frac{[L]/[T]^2 \cdot [L] \cdot [L]}{[L]/[T]} = \frac{[L]^2}{[T]}$$

checking: $\nabla\phi =$ speed $\Rightarrow$ $[\phi] = [\text{speed}] \cdot [L]$

$$[\nabla\phi] = \frac{[\phi]}{[L]}$$

$$\underbrace{\left[\frac{g\lambda a}{c_0}\right]}_{A} = \frac{[L]}{[T]} \cdot [L] = [\phi] \qquad \left(\frac{g\lambda a}{c_0}\right)$$

$$\phi = A\tilde{\phi}(\tilde{x}, \tilde{y}, \tilde{t})$$

$$\phi_x = A\tilde{\phi}_{\tilde{x}}\frac{d\tilde{x}}{dx} = A\tilde{\phi}_{\tilde{x}}\frac{1}{\lambda}$$

$$\phi_y = A\tilde{\phi}_{\tilde{y}}\frac{d\tilde{y}}{dy} = A\tilde{\phi}_{\tilde{y}}\frac{1}{h_0}$$

$$\Rightarrow \Delta\phi = \beta\tilde{\phi}_{\tilde{x}\tilde{x}} + \tilde{\phi}_{\tilde{y}\tilde{y}} = 0$$