– Substitute in the potential theory equations and drop the tilde ~ :

$\beta \phi_{xx} + \phi_{yy} = 0, \qquad -h\left(\frac{x}{\gamma}\right) < y < 1 + \alpha\eta$	
$\left.\begin{array}{l} \eta_t + \alpha \phi_x \eta_x - \frac{1}{\beta}\phi_y = 0 \\ \eta + \phi_t + \frac{1}{2}\alpha \phi_x^2 + \frac{1}{2}\frac{\alpha}{\beta}\phi_y^2 = 0 \end{array}\right\} \; y = 1 + \alpha\eta$	free surface conditions
$\phi_y + \frac{\beta}{\gamma} h'\left(\frac{x}{\gamma}\right)\phi_x = \boxed{\frac{d\phi}{dn} = 0}$ $\qquad y = -h\left(\frac{x}{\gamma}\right)$	bottom boundary condition
$\begin{cases} \phi(x, 1+\alpha\eta, 0) = \phi_0(x) \\ \eta(x,0) = \eta_0(x) \end{cases}$	initial conditions

$\alpha = \frac{a}{h_0}$ = nonlinearity parameter

$\beta = \frac{h_0^2}{\lambda^2}$ = dispersion parameter (long waves/shallow water when small)

$\gamma = \frac{l}{\lambda}$ = wave/bottom interaction parameter.

Note: when $\boxed{\alpha = 0}$ $\Longrightarrow$ no nonlinear terms in PDE
$\Longrightarrow$ not a free boundary problem anymore $\to$ "another source of nonlinearity"

— $\boxed{\text{REDUCED MODELING}}$

— systematic simplification of PDEs.

{ what happens in the limit $\alpha, \beta \to 0$?
" " " the regime $\alpha = \Theta(\varepsilon), \beta = \Theta(\varepsilon), \varepsilon \ll 1$?

— First exercise in this direction. For simplicity take a FLAT BOTTOM.

shallow waters/long waves $\Rightarrow$ $\boxed{\beta \ll 1}$

weakly nonlinear waves $\boxed{0 < \alpha \ll 1.}$ $\Rightarrow$ ◪1

in Shallow water regime the vertical structure of solution changes very little.

Lets make an expansion about the bottom in the form.
(y=0) – flat

$$\boxed{\phi(x,y,t) = \sum_{n=0}^{\infty} y^n f_n(x,t)}$$

Lets first solve laplace's equation and the (bottom) Neumann condition in this power series form!

Laplace:

recurrence relation

$$\boxed{\beta\, \phi_{xx} + \phi_{yy}} = \sum_{m=0}^{\infty} y^m \boxed{\left(\beta\, \partial_x^2 f_m + (m+2)(m+1) f_{m+2}\right)} = 0$$

$$\Rightarrow \phi(x,y,t) = \left[\sum_{n=0}^{\infty} (-\beta)^n y^{2n} \frac{\partial_x^{2n} f_0}{(2n)!}\right] + \left[\sum_{n=0}^{\infty} (-\beta)^n y^{2n+1} \frac{\partial_x^{2n} f_1}{(2n+1)!}\right]$$

even terms odd terms

Neumann Cond.

$$\boxed{\phi_y(x,0,t) \equiv 0} \Rightarrow f_1(x,t) \equiv 0$$

$\otimes_1$

$$\boxed{\phi(x,y,t) = \sum_{n=0}^{\infty} (-\beta)^n \frac{y^{2n}}{(2n)!} \partial_x^{2n} f(x,t)}$$

Satisfies Laplace + Neumann

need no index.

— Now we need to deal with the free surface conditions which are to be solved at

$$\boxed{y = 1 + \alpha\eta}$$

from $\otimes_1$

$$\begin{cases} \phi_x = f_x - \frac{\beta}{2!}(1+\alpha\eta)^2 f_{xxx} + \frac{\beta^2}{4!}(1+\alpha\eta)^4 f_{xxxxx} - \cdots \\ \phi_y = -\beta(1+\alpha\eta) f_{xx} + \frac{\beta^2}{6}(1+\alpha\eta)^3 f_{xxxx} - \cdots \\ \phi_t = f_t - \frac{\beta}{2}(1+\alpha\eta)^2 f_{xxt} + \frac{\beta^2}{4!}(1+\alpha\eta)^4 f_{xxxxt} - \cdots \end{cases}$$

— Substituting in the Bernoulli law and kinematic condition:

$$\begin{cases} \eta_t + \alpha f_x \eta_x + (1+\alpha\eta) f_{xx} - \frac{\beta}{6} f_{xxxx} = \boxed{\mathcal{O}(\beta^2, \alpha\beta, \alpha^2)} \\ \eta + f_t - \frac{\beta}{2} f_{xxt} + \frac{\alpha}{2} f_x^2 = \boxed{\mathcal{O}(\beta^2, \alpha\beta, \alpha^2)} \end{cases}$$

terms we will discard.

therefore called

weakly nonlinear
weakly dispersive
REGIMES
leading order in α & β.

– As our first approximation we have
(dropping quadratic terms in α and β)

$$\begin{cases} \eta_t + [(1+\alpha\eta) f_x]_x - \frac{\beta}{6} f_{xxxx} = 0 \\ \eta + f_t + \frac{\alpha}{2} f_x^2 - \frac{\beta}{2} f_{xxt} = 0 \end{cases}$$

$$\eta_x + \tilde{u}_t + \frac{\alpha}{2}(u^2)_x - \frac{\beta}{2}\tilde{u}_{xxt} = 0$$

leading order terms in α and β.

What is $f(x,t)$? Physical meaning? We will replace it below.

let $\boxed{\tilde{u}(x,t) \equiv f_x(x,t)}$.

slip velocity at the bottom

We used in: $\phi_x(x,Y,t) = f_x(x,t) - \frac{\beta}{2} Y^2 f_{xxx}(x,t) + O(\beta^2)$

where $\boxed{Y \equiv 1 + \alpha\eta(x,t)}$ is the free surface position.

Note that if we DEPTH AVERAGE (see comment 1 page 74)

$$\boxed{\frac{1}{Y}\int_0^Y \phi_x(x,y,t)\,dy} = \frac{1}{Y}\sum_{n=0}^{\infty} \frac{(-\beta)^n}{(2n)!}\left[\int_0^Y y^{2n}\,dy\right] \partial_x^{2n+1} f =$$

$$= \sum_{n=0}^{\infty} \frac{(-\beta)^n}{(2n)!}\,\frac{Y^{2n}}{2n+1}\,\partial_x^{2n+1} f$$

depth averaged horizontal speed

$$\boxed{\overline{U}(x,t)} = \tilde{u}(x,t) - \frac{\beta}{6}(1+\alpha\eta)^2 f_{xxx} + O(\beta^2) =$$

$$= \boxed{\tilde{u}(x,t) - \frac{\beta}{6}\tilde{u}_{xx}(x,t) + O(\beta^2, \alpha\beta)}$$

We want to invert roles in order to use/sub. in PDEs.

We have that $\boxed{\tilde{u} = u + O(\beta)}$ and

$$\boxed{\tilde{u}(x,t) = u(x,t) + \frac{\beta}{6} u_{xx}(x,t) + O(\beta^2, \alpha\beta)}$$

– Substituting in the PDEs and dropping terms of order $O(\beta^2, \alpha\beta, \alpha^2)$:

$$\begin{cases} \eta_t + \left[(1+\alpha\eta)\left(u + \frac{\beta}{6}u_{xx}\right)\right]_x - \frac{\beta}{6} u_{xxx} = 0 \\ \left[u + \frac{\beta}{6}u_{xx}\right]_t + \eta_x + \alpha\left\{\left[u + \frac{\beta}{6}u_{xx}\right]\left[u_x + \frac{\beta}{6}u_{xxx}\right]\right\} - \frac{\beta}{2} u_{xxt} = 0 \end{cases}$$

Note that we have differentiated this equation with respect to x in order to have speeds and not reduced potentials $f(x,t)$.

– Dropping higher order terms:

$$\begin{cases} \eta_t + \left[(1+\alpha\eta)u\right]_x = 0 \\ u_t + \eta_x + \alpha u u_x - \frac{\beta}{3} u_{xxt} = 0 \end{cases}$$

– From the first equation we can use that $\boxed{\eta_t = -u_x + O(\alpha)}$ which allows to exchange x- and t-derivatives:

$$\boxed{\begin{aligned} &\eta_t + \left[(1+\alpha\eta)u\right]_x = 0 \\ &u_t + \eta_x + \alpha u u_x + \frac{\beta}{3}\eta_{xtt} = 0 \end{aligned}}$$

Boussinesq system.

dimensionless form similar to original form

weakly nonlinear

weakly dispersive

↪

– In DIMENSIONAL variables the original system was given as

original dimensional $\boxed{\text{Boussinesq equations (1872)}}$

$$\begin{cases} H_t + (uH)_x = 0 \\ u_t + gH_x + uu_x + \frac{1}{3} h_0 H_{xtt} = 0 \end{cases}$$

u = horizontal speed (average)

$H(x,t)$ = total water depth

$h_0 + \eta(x,t)$

Remark 1 : $\boxed{\beta = 0 \text{ regime}}$

non dispersive regime

shallow water system (page 42)

$$\begin{cases} H_t + (uH)_x = 0 \\ u_t + \left(gH + \frac{u^2}{2}\right)_x = 0 \end{cases}$$

or

$$\begin{cases} \eta_t + ((1+\alpha\eta)u)_x = 0 \\ u_t + \left(\eta + \alpha\frac{u^2}{2}\right)_x = 0 \end{cases}$$

dimensionless

$\boxed{\alpha = 0 + \beta = 0}$ linear hyperbolic

$\tilde{u} = u + \frac{\beta}{6} u_{xx} + O(\beta^2, \alpha\beta)$

$$\left.\begin{aligned} \eta_t + u_x &= 0 \\ u_t + \eta_x &= 0 \end{aligned}\right\} \quad \boxed{\eta_{tt} - \eta_{xx} = 0}$$

Wave equation.

— How to extract a unidireccional weakly nonlinear, weakly dispersive wave equation (namely the KdV) from the Boussinesq system?

— Go back to version in page 39:

$$\Phi_1 \quad \begin{cases} \eta_t + [(1+\alpha\eta)\tilde{u}]_x - \frac{\beta}{6}\tilde{u}_{xxx} = \mathcal{O}(\alpha^2, \alpha\beta, \beta^2) \\ \tilde{u}_t + \alpha\tilde{u}\tilde{u}_x + \eta_x - \frac{1}{2}\beta\tilde{u}_{xxt} = \mathcal{O}(\alpha^2, \alpha\beta, \beta^2) \end{cases}$$

— In the linear case we know that we can choose "uni-directional"-data for

$$\begin{cases} \eta_t + \tilde{u}_x = 0 \\ \tilde{u}_t + \eta_x = 0 \end{cases}$$

Let $\boxed{\tilde{u}_0 = \eta_0} \Rightarrow \begin{cases} \eta_x + \eta_t = 0 \Rightarrow \eta(x,t) = \eta_0(x-t) \quad \text{(CHECK!)} \\ \text{and } \boxed{\eta \equiv u} \leftarrow \end{cases}$

— this provides us with an ansatz: $\boxed{\underbrace{\boxed{\tilde{u} = \eta}}_{\text{ansatz}\ \otimes_1} + \underbrace{\alpha A}_{\otimes_2} + \underbrace{\beta B}_{\otimes_3} + \mathcal{O}(\alpha^2 + \beta^2)}$ Φ_2

$\otimes_1$: what we learned from the linear problem

$\otimes_2$: $A = A(x,t)$, a weakly nonlinear correction to be found.

$\otimes_3$: $B = B(x,t)$, a weakly dispersive corretion to be found.

Plugging Φ_2 into Φ_1 one can find to LEADING ORDER* that

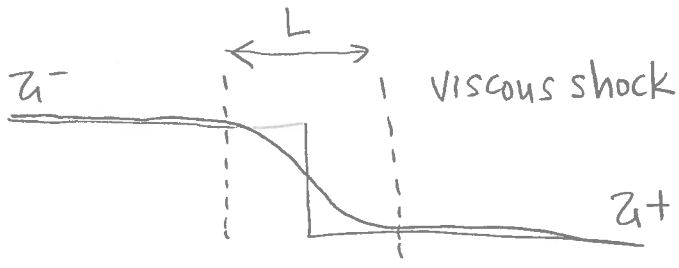
KdV $\boxed{\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{\beta}{6}\eta_{xxx} = 0}$

* dropping $\mathcal{O}(\alpha^2, \alpha\beta, \beta^2)$ terms.

(1) $u_t + u u_x = 0$ Burgers $\equiv$ hyperbolic

(2) $u_t + u u_x = \varepsilon u_{xx}$ (viscous) Burgers $\equiv$ advection-diffusion

(3) $u_t + u u_x + \varepsilon u_{xxx} = 0$ KdV $\equiv$ dispersive

(2) 

L scales like

$$\underbrace{u(x-ct)}_{\xi = x-ct} = \frac{u^+ + u^-}{2} - \left(\frac{u^+ - u^-}{2}\right) \tanh\left(\left(\frac{u^- - u^+}{4\varepsilon}\right)\xi\right)$$

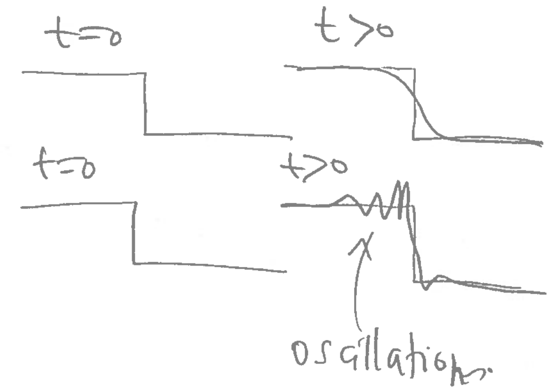
(3) $\eta_t + 6\eta\eta_x + \eta_{xxx} = 0$

$$\eta(x,t) = f(x-ct) = \frac{1}{2} c \operatorname{sech}^2\left[\frac{\sqrt{c}}{2}(x - ct - x_0)\right]$$

Solitary wave

linear case $u_t + c u_x = \varepsilon u_{xx}$

$u_t + c u_x + \varepsilon u_{xxx} = 0$



— LINEAR DISPERSION (Fourier) ANALYSIS —

— Take the Boussinesq system (dimensionless) with

$\alpha = \beta = 0$
$$\#\quad \begin{cases} \eta_t + u_x = 0 \\ u_t + \eta_x = 0 \end{cases}$$

and lets do a DISPERSION ANALYSIS in order to get the dispersion relation : $\omega = \omega(k)$. Recall $\omega = \frac{2\pi}{T}$, $k = \frac{2\pi}{\lambda}$.

Let

$$\#_2\quad \begin{cases} \eta(x,t) = \displaystyle\int_{-\infty}^{\infty} \hat{\eta}_1(k)\, \underbrace{e^{i(kx-\omega(k)t)}}_{\text{right propag. mode}} dk + \int_{-\infty}^{\infty} \hat{\eta}_2(k)\, \underbrace{e^{i(kx+\omega(k)t)}}_{\text{left propag. mode}} dk \\ u(x,t) = \displaystyle\int_{-\infty}^{\infty} \hat{u}_1(k)\, e^{i(kx-\omega(k)t)} dk + \int_{-\infty}^{\infty} \hat{u}_2(k)\, e^{i(kx+\omega(k)t)} dk \end{cases}$$

Fourier amplitudes

The problem is linear. Therefore we can consider a mode at a time. Take a generic mode, substitute in $\#$ and get

$$\begin{cases} -i\omega\, \hat{\eta}_1 + ik\, \hat{u}_1 = 0 \\ -i\omega\, \hat{u}_1 + ik\, \hat{\eta}_1 = 0 \end{cases}.$$

This system has a nontrivial solution (for the Fourier amplitudes) if $\begin{vmatrix} -i\omega & ik \\ ik & -i\omega \end{vmatrix} = 0$

We have therefore the dispersion relation

$$\omega^2 = k^2 \Rightarrow \boxed{\omega(k) = \pm k}$$

note that these have already been taken into account in $\#_2$

The system being hyperbolic we have that all Fourier modes travel with the same $\boxed{\text{PHASE SPEED } C(k) = \frac{\omega(k)}{k}}$

Here $\boxed{C(K) = \pm 1}$

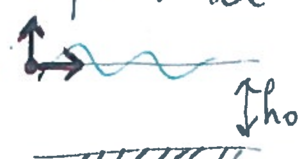
$$\begin{cases} \eta(x,t) = \int_{-\infty}^{\infty} \hat{\eta}_1(k)\, e^{ik(x-t)}\, dk + \int_{-\infty}^{\infty} \hat{\eta}_2(k)\, e^{ik(x+t)}\, dk \\ u(x,t) = \int_{-\infty}^{\infty} \hat{u}_1(k)\, e^{ik(x-t)}\, dk + \int_{-\infty}^{\infty} \hat{u}_2(k)\, e^{ik(x+t)}\, dk . \end{cases}$$

This is not the case in potential theory, where the system is DISPERSIVE as we will see.

How are $\hat{\eta}_1, \hat{\eta}_2, \hat{u}_1$ and $\hat{u}_2$ computed?

LINEAR Potential Theory

– Lets start with the $\boxed{\text{linear}}$ DIMENSIONAL potential theory equations:



does not really matter where origin is

$$\phi_{xx} + \phi_{yy} = 0 \qquad -h_0 < y < 0$$

$$\phi_{tt} + g\,\phi_y = 0 \qquad y = 0$$

$$\phi_y = 0 \qquad y = -h_0 .$$

The wave elevation becomes a "passive" variable in our system since we find it after the potential problem has been solved:

$$\boxed{\eta(x,t) = \frac{-1}{g}\, \phi_t(x,0,t)}$$

— Take the Fourier representation for the potential:

$$\phi(x,y,t) = \int_{-\infty}^{\infty} \hat{\Phi}_1(y,k)\, e^{i(kx-\omega t)}\, dk + \int_{-\infty}^{\infty} \hat{\Phi}_2(y,k)\, e^{i(kx+\omega t)}\, dk .$$

— We want that these are harmonic functions satisfying the Neumann condition at the bottom. Or in other words, we want to find the harmonic extension of a Fourier mode given along the (undisturbed) free surface:

$$\begin{cases} \dfrac{d^2\hat{\phi}_i}{dy^2} - k^2\hat{\phi}_i = 0 , \quad i=1\,\&\,2, & \Rightarrow \begin{cases} \cosh(k(y+h_0)) \\ \sinh(k(y+h_0)) \end{cases} \\ \dfrac{d\hat{\phi}_i}{dy} = 0 , \quad \text{at } y=-h_0 . \end{cases}$$

Using the Neumann Cond:

$$\boxed{\hat{\phi}_i(y,k) = F_i(k)\cosh(k(y+h_0))}$$

Now we need to satisfy the "free surface" condition

$$\boxed{\phi_{tt} + g\phi_y = 0}$$

$$\hookrightarrow \quad -\omega^2 F_i \cosh(kh_0) + kg F_i \sinh(kh_0) = 0$$

which leads to the DISPERSION RELATION

$$\boxed{\omega^2(k) = kg\tanh(kh_0)}$$

- Substituting back yields

right going → ← left going

$$\boxed{\phi(x,y,t) = \int_{-\infty}^{\infty} F_1(k)\cosh(k(y+h_0))\, e^{ik(x-c(k)t)}\, dk + \int_{-\infty}^{\infty} F_2(k)\cosh(k(y+h_0))\, e^{ik(x+c(k)t)}\, dk}$$

where the PHASE SPEED is

$$\boxed{c(k) \equiv \frac{\omega(k)}{k}} \quad , \quad c^2(k) = \frac{g}{k}\tanh(kh_0)$$

- Each Fourier mode has a phase speed that is a wave number-dependent. Long modes travel faster than shorter modes. This k-dependent phase speed characterizes a DISPERSIVE SYSTEM.

- Two limits
  - SHALLOW WATERS - LONG WAVES

$$\lim_{k\to 0} c(k) = \lim_{k\to 0} \sqrt{\frac{g}{k}\tanh(kh_0)} = \boxed{\sqrt{gh_0}} \quad \leftarrow \text{k-independent}$$

  - DEEP WATERS - SHORT WAVES

$$\boxed{\lim_{h_0\to\infty} c(k) = \boxed{\sqrt{\frac{g}{k}}}}$$

Dimensionless version

$$\boxed{\omega^2 = \frac{k}{\sqrt{\beta}}\tanh k\sqrt{\beta}}$$

$\beta \to 0$

$\beta \to \infty$

$g \cong 10\, m/s^2 \quad h_0 = 4000 m$

$\sqrt{gh_0} = 200\, m/s = 720\, km/h$

— Take the system in page 76 (with ∂_x · second eqn.)

$$\begin{cases} \eta_t + [(1+\alpha\eta)\tilde{u}]_x - \frac{\beta}{6}\tilde{u}_{xxx} = 0 \\ \tilde{u}_t + \eta_x + \alpha\tilde{u}\tilde{u}_x - \frac{\beta}{2}\tilde{u}_{xxt} = 0 \end{cases} \underset{\alpha=0}{\Rightarrow} \begin{cases} \eta_t + \tilde{u}_x - \frac{\beta}{6}\tilde{u}_{xxx} = 0 \\ \tilde{u}_t + \eta_x - \frac{\beta}{2}\tilde{u}_{xxt} = 0 \end{cases}$$

$$\Rightarrow \det \begin{bmatrix} ik - \frac{\beta}{6}(-ik^3) & -i\omega \\ -i\omega - \frac{\beta}{2}(k^2 i\omega) & ik \end{bmatrix} = 0$$

$$-k^2 - \frac{\beta}{6}k^4 + \omega^2 + \omega^2\frac{\beta}{2}k^2 = 0$$

$\tilde{u}$ = velocity at bottom

Dispersion relation — weakly dispersive

$$\boxed{\omega^2 = \frac{k^2\left(1+\beta/6\, k^2\right)}{1+\beta/2\, k^2}} \approx \left(k^2 + \frac{\beta}{6}k^4\right)\left(1 - \frac{\beta}{2}k^2 + O(\beta^2)\right),$$

$$= k^2 + \left(\frac{\beta}{6} - \frac{\beta}{2}\right)k^4 + \dots$$

$$\boxed{\omega^2 \approx k^2 - \frac{\beta}{3}k^4 + O(\beta^2)}.$$

"Full" dispersion relation:

$$\boxed{\boxed{\omega^2 = \frac{k}{\sqrt{\beta}}\tanh(k\sqrt{\beta})} \approx k^2 - \frac{\beta}{3}k^4 + O(\beta^2)}.$$

$$\boxed{\omega^2(k) = \frac{p(k)}{q(k)} = \frac{k^2 + \beta/6\, k^4}{1 + \beta/2\, k^2}}$$

Padé aproximation of

– What can you say of a system like

$$\begin{cases} \eta_t + u_x = 0 \\ u_t + \eta_x + \frac{\beta}{3}\eta_{xxx} = 0 \end{cases}$$

$$\det \begin{bmatrix} -i\omega & ik \\ ik\left(1-\frac{\beta}{3}k^2\right) & -i\omega \end{bmatrix} = 0 \quad \Longrightarrow \quad \boxed{\omega^2 = k^2\left(1 - \frac{\beta}{3}k^2\right)}$$

$k > \sqrt{3/\beta} \Rightarrow$ complex freq.

– linear KdV (page 79)

$$\boxed{\eta_t + \eta_x + \frac{\beta}{6}\eta_{xxx} = 0}$$

$$-i\omega + ik + \frac{\beta}{6}(ik)^3 = 0$$

$$-i\omega + ik - i\frac{\beta}{6}k^3 = 0 \qquad \boxed{\omega = k - \frac{\beta}{6}k^3}$$

$$\boxed{\frac{\omega}{k} = 1 - \frac{\beta}{6}k^2}$$

– Benjamin-Bona-Mahony (BBM) '1972, Philos. Trans. Roy. Soc. London, Series A

(KdV) $\eta_t + \eta_x + \eta\eta_x + \eta_{xxx} = 0$

(BBM) $\eta_t + \eta_x + \eta\eta_x - \eta_{xxt} = 0$

↑ change this derivative with the trick we saw.

— linear BBM

$$\eta_t + \eta_x - \eta_{xxt} = 0$$

$$-i\omega + ik - (-k^2)(-i\omega) = 0$$

$$-i\omega(1 + k^2) = -ik$$

$$\boxed{\omega = \frac{k}{1+k^2}}$$ BBM

$k \ll 1$

$\omega \sim k(1-k^2)$

KdV

$$\boxed{\omega = k - k^3}$$

$$\boxed{\frac{\omega}{k} = \frac{1}{1+k^2}}$$ ← non-physical (ie. mathematical) high fequencies barely propagate

— BBM also known as the regularized long-wave equation

↳ due to cross derivative (η_{xxt}) ⟶ see 86A-B

⊙ Facts

— Both KdV and BBM have sech^2 – solitary wave solutions (nonlinear traveling waves)

— Solitary wave ≡ pulse shaped – single hump as described by Scott Russell, 1834 which called traveling wave a wave of translation

— KdV – called an integrable system.

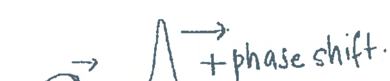
— has infinitely conserved quantities (Miura, Gardner & Kruskal '1968)

— solitary waves are solitons – collide in an "elastic fashion"

— profile unchanged

— no radiation

2 solitary waves



profiles unchanged.

- BBM has 3 conservation laws

  - Solitary waves are NOT solitons

    after interaction with other solitary waves an oscillatory tail is generated and the solitary wave profile changes (a bit).

- Boussinesq system (page 77)

$$\begin{cases} \eta_t + \left[(1+\alpha\eta)u\right]_x = 0 \\ u_t + \eta_x + \alpha u u_x + \frac{\beta}{3}\eta_{xtt} = 0 \end{cases}$$

$\boxed{\alpha = 0}$

$$\det \begin{bmatrix} -\omega & k \\ k - \frac{\beta}{3}k\omega^2 & -\omega \end{bmatrix} = 0 \Rightarrow \boxed{\omega^2 = \frac{k^2}{1+\frac{\beta}{3}k^2}}$$

$$\boxed{\frac{\omega}{k} = \pm\sqrt{\frac{1}{1+\frac{\beta}{3}k^2}}}$$