Leading order:

$$\phi_x = f_x - \frac{\beta}{2}(1+\alpha\eta)^2 f_{xxx} + \overbrace{O(\beta^2)}^{O(\varepsilon^2)}$$

$$\phi_y = -\beta(1+\alpha\eta) f_{xx} + \underbrace{O(\beta^2)}_{O(\varepsilon^2)}$$

we only want to keep linear terms in α and β. I.e truncate stuff beyond $O(\alpha^2, \alpha\beta, \beta^2)$.

Put this in (2) and (3) and truncate

$$\begin{cases} \eta_t + \alpha f_x \eta_x + (1+\alpha\eta) f_{xx} - \frac{\beta}{6} f_{xxxx} = 0 & (2) \\ \eta + f_t + \frac{\beta}{2} f_{xxt} + \frac{\alpha}{2} f_x^2 = 0 & (3) \end{cases}$$

Boussinesq equations: η and $u \sim$ average of ϕ over y.

$$\frac{1}{Y}\int_0^Y \phi_x(x,y,t)\,dy \equiv U(x,t)$$

Take -4 -

$$\begin{cases} h_t + (hu)_x = 0 \\ u_t + u u_x + h_x = 0 \end{cases} \iff \begin{pmatrix} h \\ u \end{pmatrix}_t + \overbrace{\begin{pmatrix} u & h \\ 1 & u \end{pmatrix}}^{U} \begin{pmatrix} h \\ u \end{pmatrix}_x = 0$$

In the linearized system:

$$\begin{cases} h_t + u_x = 0 \\ u_t + h_x = 0 \end{cases} \Rightarrow \begin{cases} (h+u)_t + (\overbrace{h+u}^{R})_x = 0 \\ (\underbrace{h-u}_{L})_t - (h-u)_x = 0 \end{cases}$$

Since $R_T + R_x = 0$, $\frac{dx}{dt} = 1 \Rightarrow \frac{dR}{dt} = 0$.

$\hat{\text{L}}$ Characteristic method.

$$L_t - L_x = 0, \quad \frac{dx}{dt} = -1 \Rightarrow \frac{dL}{dt} = 0.$$

Call $A = \begin{pmatrix} u & h \\ 1 & u \end{pmatrix}$, $U = \begin{pmatrix} h \\ u \end{pmatrix}$.

$$U_t + A U_x = 0 \Rightarrow \underbrace{(l_1 \ \ l_2)}_{l} U_t + (l_1 \ \ l_2) A U_x = 0.$$

If $lA = \lambda l$ (left eigenvector and eigenvalues) we get

$$l U_t + \lambda l U_x = 0$$

Since A depends on u and h we λ and l also do.

If l is constant,

$(lU)_t + \lambda (lU)_x = 0$ and it would be easy to solve again with characteristic.

Consider $A = \begin{pmatrix} u & h \\ 1 & u \end{pmatrix}$.

$$0 = \det(A - \lambda I) = \lambda^2 - 2u\lambda + (u^2 - h) = 0$$

$\Rightarrow \lambda = u \pm \sqrt{h}$ (This is velocity of wave + velocity of water inside the wave).

We now get corresponding eigenvectors

$$l_1 = \pm \frac{1}{\sqrt{h}}, \quad l_2 = 1, \quad \lambda = u \pm \sqrt{h}.$$

Plugging l_1, l_2, λ we get the equations

$$\pm \frac{h_t}{\sqrt{h}} + u_t + (u \pm \sqrt{h})\left(\pm \frac{h_x}{\sqrt{h}} + u_x\right) = 0$$

We have $\frac{h_t}{\sqrt{h}} = (2\sqrt{h})_t$

$$(\overbrace{u \pm 2\sqrt{h}}^{R^\pm})_t + (\overbrace{u \pm \sqrt{h}}^{\lambda^\pm})(\overbrace{u \pm 2\sqrt{h}}^{R^\pm})_x = 0$$

This is exactly the original problem in 1D except $\lambda^\pm$ is not a function of solely $R^\pm$. Indeed

$$\lambda^+ = \Gamma^+(R^+, R^-)$$
$$\lambda^- = \Gamma^-(R^+, R^-)$$

Characteristics are not straight lines at the beginning in general because $\lambda^\pm$ does

Consider now the Riemann problem

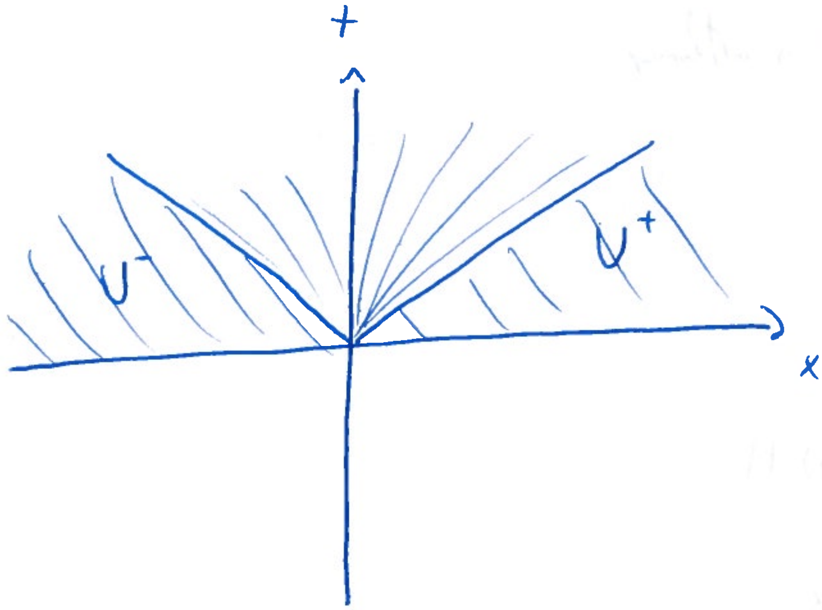
$$\begin{cases} U_t + AU_x = 0 \\ U(x,0) = \begin{cases} U^+ & \text{if } x > 0 \\ U^- & \text{if } x < 0 \end{cases} \end{cases}$$

By the same symmetry argument $U(x,t) = U(\overset{\xi}{\overset{\parallel}{\tfrac{x}{t}}})$. Then

$U_t = -\frac{x}{t^2} U'$, $U_x = \frac{1}{t} U'$, so equation becomes

$$(-\xi I + A) U' = 0$$
$$\Downarrow$$

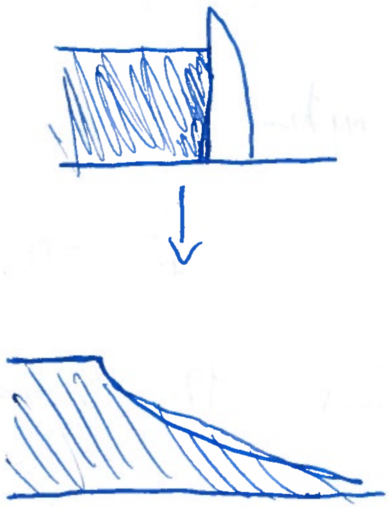
$U' = 0$ or U' is eigenvector of A w eigenvalue ξ.



Two fans corresponding to λ_+ and λ_-.

In the "first" R^+ is const and in the "second" R^- is const.

<u>Problem:</u> Imagine we have water with an obstacle that breaks. Describe what happens.



<u>Examples</u>

$u_{tt} - u_{xx} = 0$ Wave → Has solutions e.g. $\cos(kx - \omega t)$ if $\omega^2 = k^2$ ⇝ Dispersion equation.

$i\Psi_t = \Psi_{xx}$ Schrödinger

↳ Schrödinger has solution $\Psi = e^{i(kx-\omega t)}$ if $\omega = -k^2$

Consider a solution of the form

$a(x,t)e^{i\theta(x,t)}$ Locally ($a(x,t)$ is const (?)) we should have $\omega = \Omega(k,x,t)$ where

$k = \theta_x$, $\omega = -\theta_t$ (motivation from the above example). Then we have $k_t = \theta_{xt} = \theta_{tx} = -\omega_x \Rightarrow k_t + \omega_x = 0$.

Now $\omega = \Omega(k,x,t)$ we get something

$$k_t + \Omega_k k_x + \Omega_x = 0.$$

We now use characteristics. If

$$\frac{dx}{dt} = \Omega_k \, , \quad \text{since } k = k(x(t),t)$$

$$\Downarrow$$

$$\frac{dk}{dt} = k_t + \frac{dx}{dt} k_x$$

$$= k_t + \Omega_k k_x = -\Omega_x$$

Then we have the system

$$\frac{dx}{dt} = \Omega_k \qquad \frac{dk}{dt} = -\Omega_x .$$

If $\Omega \to H$, $k \to p$ this becomes

$$\begin{cases} \dot{x} = H_p \\ \dot{p} = -H_x \end{cases}$$