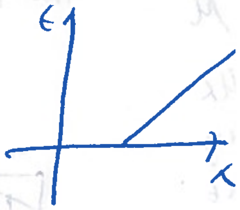
$\rho(x,t)$ $\quad \rho_t + c(\rho)\rho_x = 0$ $\quad \frac{dx}{dt} = c(\rho)$

$\frac{d\rho}{dt} = 0$



$e^{i(\underbrace{kx - \omega t}_{\theta})}$ $\quad \omega = \Omega(k)$

This relation may depend on where you are. $\quad \omega = \Omega(k, x, t)$ $\quad \theta(x,t)$

$k = \theta_x$

$\omega = -\theta_t$

$k_t + \omega_x = 0$

$k_t + \Omega_k k_x + \Omega_x = 0$

$$\begin{cases} \frac{dx}{dt} = \Omega_k \\ \frac{dk}{dt} = -\Omega_x \end{cases}$$

$\theta = k(x - \frac{\omega}{k} t)$ $\quad \frac{\omega}{k}$ ← Phase veloc

Information travels at a different speed. $\quad \frac{\partial \omega}{\partial k}$ ← Group Velocity

$\omega \longleftrightarrow E = \frac{1}{2}mv^2$

$k \longleftrightarrow p = mv$

$E \sim p^2$

$\omega = k^2$

$\boxed{i\psi_t + \psi_{xx} = 0}$

Schödinger

$c_{ph} = \frac{\omega}{k} = k \qquad c_g = \frac{\partial \omega}{\partial k} = 2k$

## More dimensions

$\rho_t + c_x^{(\rho)}\rho_x + c_y^{(\rho)}\rho_y = 0$ $\quad \frac{dx}{dt} = c_x \quad \frac{dy}{dt} = c_y \quad \frac{d\rho}{dt} = 0$

$\theta(x,y,t)$ $\quad k = \theta_x \quad l = \theta_y$

$\omega = -\theta_t$

$k_t + \omega_x = 0$

$l_t + \omega_y = 0$

$\omega = \Omega(k, l, x, t)$

$\frac{dx}{dt} = \Omega_k \quad \frac{dk}{dt} = -\Omega_x$

$\frac{dy}{dt} = \Omega_l \quad \frac{dl}{dt} = -\Omega_y$

Can I combine both of the ideas to get a general result?

$u$

$u_x$

$u_y$

$u_t$

For this exercise let $x$ represent all variables (time included)

$x, u, p = \nabla u$

$$\boxed{F(x, u, p) = 0}$$

Not necessarily linear in $p$.

We will assume that $F$ does not depend on $u$.

$F(x, \cancel{u}, p) = 0$ First order PDE

Is there some line $x(s)$ along $s$ with I can solve the PDE

$p_k = \frac{\partial u}{\partial x_k}$ $\frac{d}{ds} x = \dot{x}$ $\dot{p}_k = \sum_l \frac{\partial^2 u}{\partial x_k \partial x_l} \dot{x}_l$

$$\frac{dF}{dx_k} = \frac{\partial F}{\partial x_k} + \sum_l \frac{\partial F}{\partial p_l} \underbrace{\frac{\partial p_l}{\partial x_k}}_{\frac{\partial^2 u}{\partial x_k \partial x_l}} = 0$$

$$\begin{cases} \dot{x}_k = \frac{\partial F}{\partial p_k} \\ \dot{p}_k = -\frac{\partial F}{\partial x_k} \end{cases}$$

$$\frac{du}{ds} := \dot{u} = \sum_l \frac{\partial u}{\partial x_l} \dot{x}_l = \sum_l p_l \frac{\partial F}{\partial p_l}$$

---

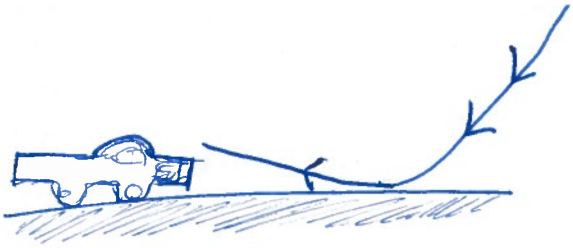
$u_{tt} - c^2 \Delta u = 0$ Wave equation in a dispersive medium $c = c($

$u(x,t) = U\left(\frac{1}{\varepsilon} \varphi(x,t)\right)$ The wave has a very rapid oscilating behaviour

$u_t = \frac{1}{\varepsilon} U' \varphi_t$ $u_{tt} = \frac{1}{\varepsilon^2} U'' \varphi_t^2 + \frac{1}{\varepsilon} U' \varphi_{tt}$

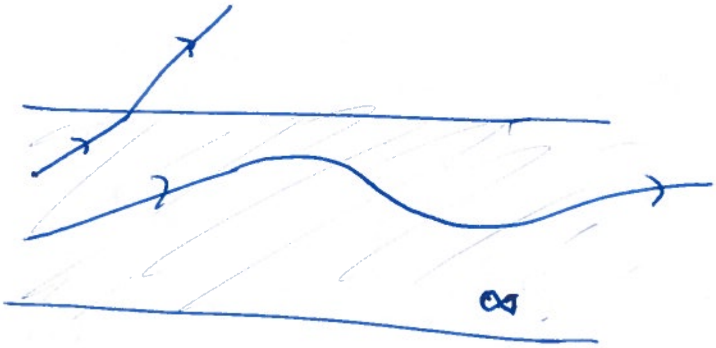
$$\varphi_t^2 - c^2(x,y)\left[\varphi_x^2 + \varphi_y^2\right] = 0$$ Eikonal Equation

It works as our $\beta$ case. The equation only depends on 1st order derivatives.



Ololocar

Ocean



There are lots of applications to this kind of thing.

$$F(u,p)=0$$

$$F(u,\nabla u)=0 \xrightarrow[\text{time}]{\text{out}} S_t + H(x_T, \nabla S) = 0$$

Hamilton-Jacobi Equation

$$i\varphi_t + \Delta\varphi = 0$$

$$|\nabla\varphi|^2$$