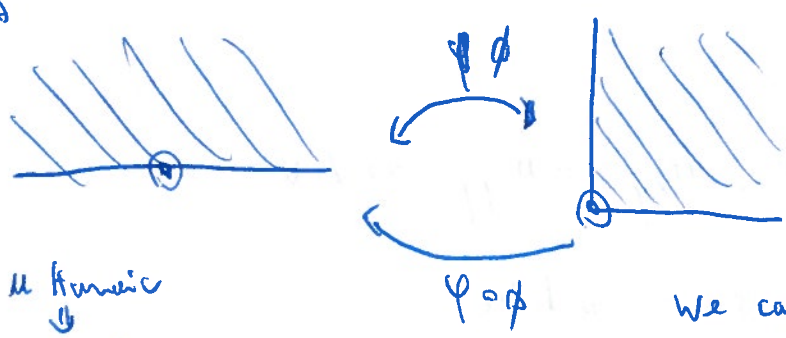
Conformal transformation

Transformations that change the domain

e.g.

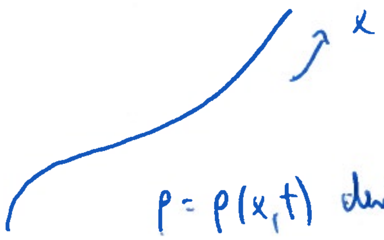


In this case $\phi : z \to z^2$.

u Harmonic
$\Downarrow$
φ analytic

We can get solutions on different domains

Taba k-2

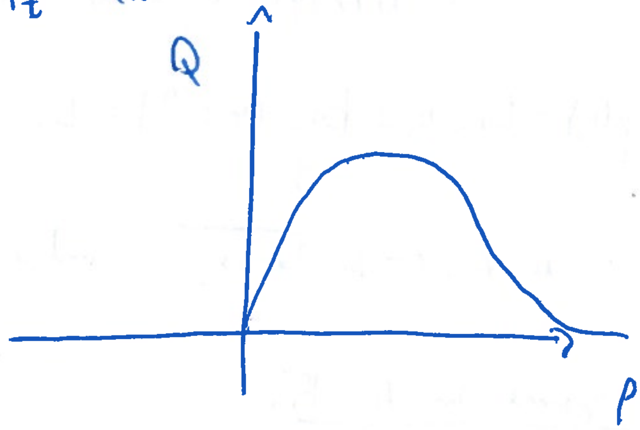


$\rho = \rho(x,t)$ density $\quad u(x,t)$ velocity $\qquad Q = \rho u.$

Conservation of mass: $\displaystyle\int_{x_1}^{x_2} \rho(x,t_2)\,dx - \int_{x_1}^{x_2} \rho(x,t_1)\,dx = \int_{t_1}^{t_2} \left[Q(x_1,t) - Q(x_2,t) \right] dt$

$$\Downarrow$$

$$\rho_t + Q_x = 0$$



We assume that $Q(x,t) = Q(\rho)$

Equation transforms in

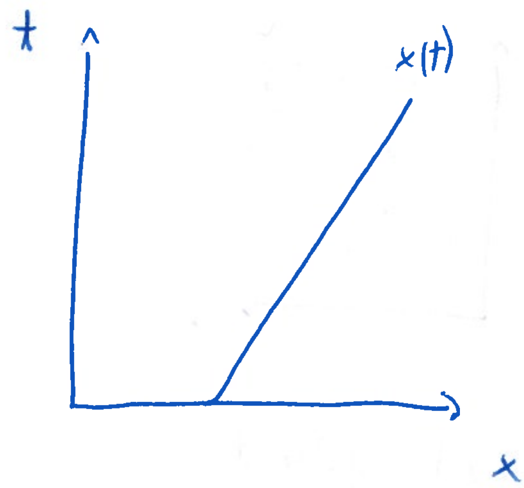
$$\boxed{\rho_t + Q'(\rho)\,\rho_x = 0}$$

Method of characteristics: Consider curve $x = x(t)$.

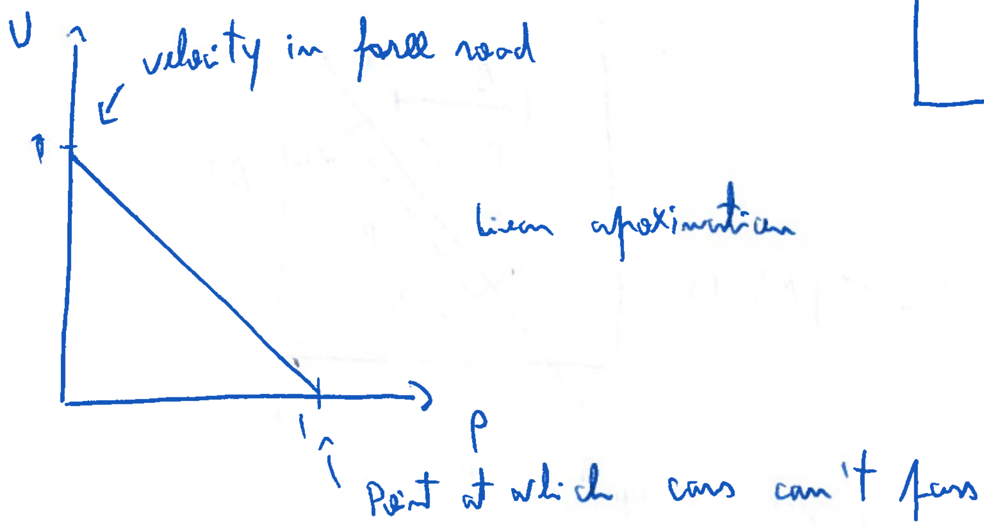
Eq. becomes

$\dfrac{dx}{dt} = Q'(\rho)$, $\quad \dfrac{d\rho}{dt} = 0 \implies x = x(t)$ is a line.

"Information travels through line $x = x(t)$"



$$Q = \rho U$$



$$U = 1-\rho, \quad Q = \rho(1-\rho)$$

Equation becomes

$$\rho_t + (\rho(1-\rho))_x = 0$$

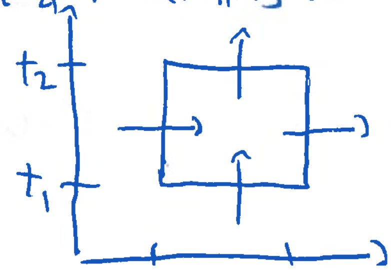
Nonlinear eq. $u_t + \left(\frac{u^2}{2}\right)_x = 0$

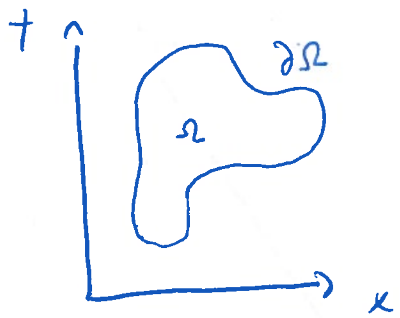
$\rho_t + Q_x = 0$, we rename $\left.\begin{array}{l} Q \to u \\ \rho \to v \\ t \to y \end{array}\right\}$ $u_x + v_y = 0$

That is, divergence of $(x,t) \to (Q,\rho)$

<u>Integral principle of conservation</u>

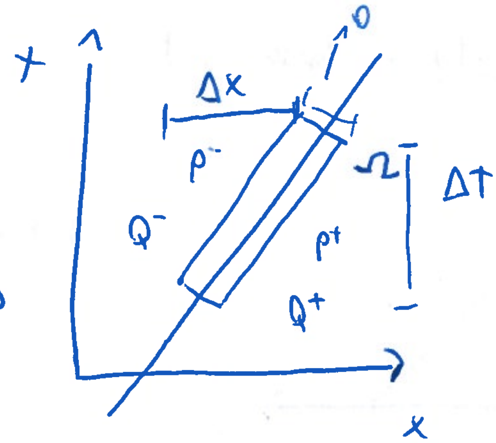
$$\int_{x_1}^{x_2} \left(\rho(x,t_2) - \rho(x,t_1)\right) dx + \int_{t_1}^{t_2} \left[Q(x_2,t) - Q(x_1,t)\right] dt = 0$$





Green theorem

$$\int_{\partial\Omega} (\rho\, dx - Q\, dt) = 0$$



$\Delta x, \Delta t$ infinitesimal.

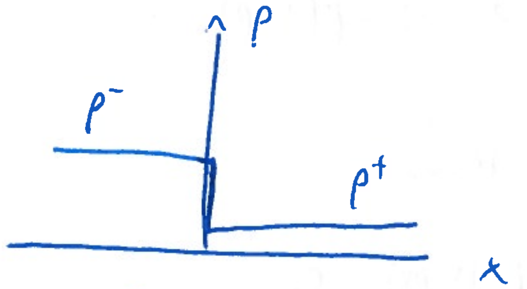
Then $(\rho^+ - \rho^-)\Delta x - (Q^+ - Q^-)\Delta t = 0$

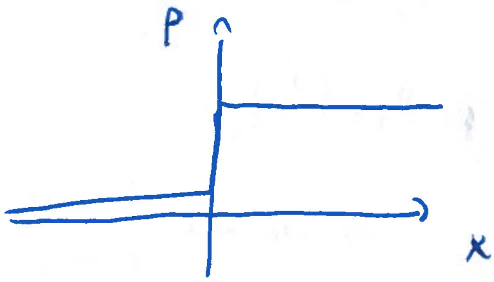
$$s = \frac{\Delta x}{\Delta t} = \frac{Q^+ - Q^-}{\rho^+ - \rho^-} = \frac{[Q]}{[\rho]}.$$

$x = x(t)$ is equation of shock.

Riemann Problem

$$\rho(x,0) = \begin{cases} \rho^- & x < 0 \\ \rho^+ & x \geq 0 \end{cases}$$





If we stretch x and t

$x' = \lambda x$ initial cond. doesn't change.

$t' = \lambda t$ equation doesn't change.

If we suppose ~~eq. doesn't~~ solution is unique:

$$\rho(\lambda x, \lambda t) = \rho(x,t).$$

ρ is constant along lines.

$$\rho(x,t) = R(\underbrace{\tfrac{x}{t}}_{\xi})$$

This means that

$$\rho_t + Q'(\rho)\rho_x = 0 \qquad Q' = c$$

$$\rho_t = -\frac{x}{t^2} R' \qquad \rho_x = \frac{1}{t} Q'.$$

Then

$$R'\left(-\frac{x}{t^2} + c(R)\cdot\frac{1}{t}\right) = 0 \Rightarrow$$

$$R'_x\left(-\xi + c(R)\right) = 0 \qquad R'(\xi)\left(-\xi + c(R(\xi))\right) = 0$$

Either $R'(\xi) = 0$ or $-\xi + c(R(\xi)) = 0$

$\Downarrow$ $\qquad\qquad\qquad \rightarrow c(R) = \xi$

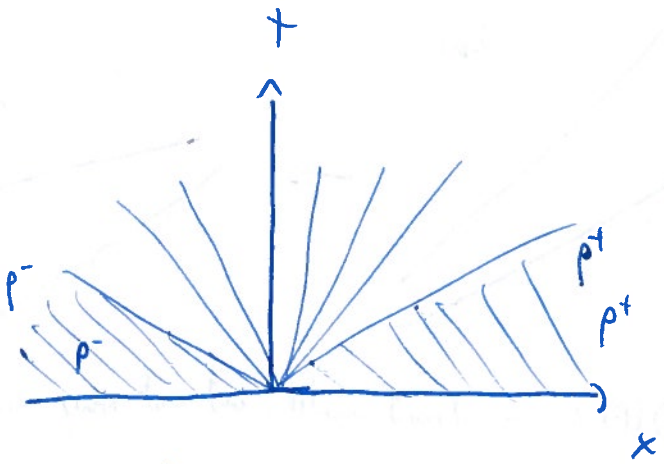
ρ constant $\qquad\qquad \overset{\shortparallel}{Q'(R)}$

$\rho^- < \rho^+$

Example: $Q = \rho(1-\rho)$, $Q' = 1 - 2\rho$

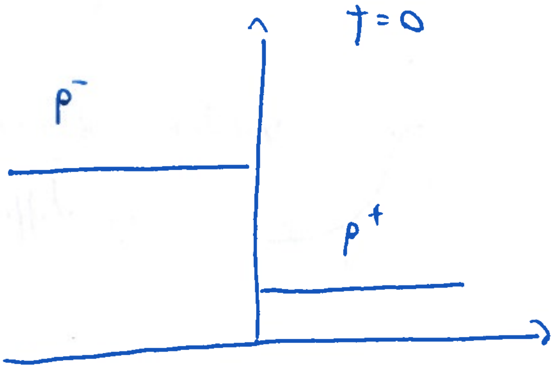
$\Downarrow$

$$\rho = \frac{1-\xi}{2} = \frac{1-\frac{x}{t}}{2}$$

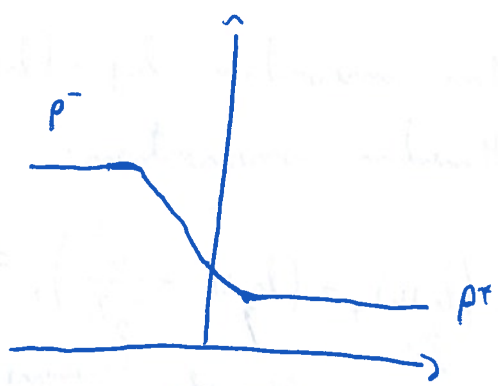


ρ is constant until a point $= \rho^-$, ~~meets~~ increases linearly until ρ^+, then stabilizes.

If $\rho^- < \rho^+$ $\qquad t = 0$

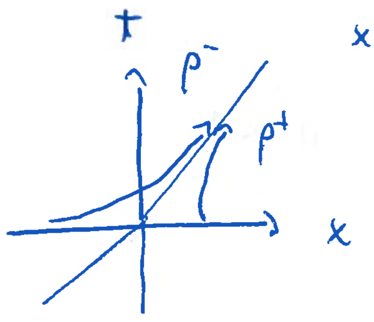


$\rightarrow$



If $\rho^- > \rho^+$, $R' = 0$

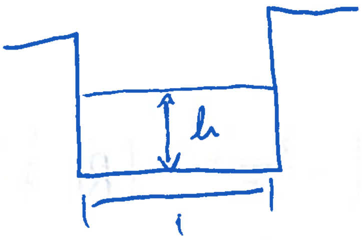
The discontinuity moves with speed $s = \frac{[Q]}{[\rho]}$



$x = st$

The characteristics converge to the shock

River



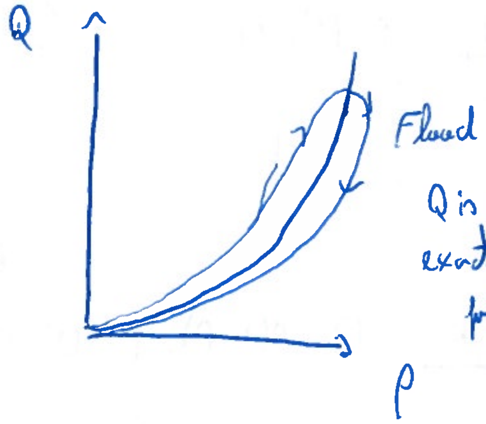
$\rho \propto h$

$\rho = h$

We have the same equation

$\rho_t + Q_x = 0$ ~~$Q = Q(\rho)$~~.

In this case U grows with ρ.

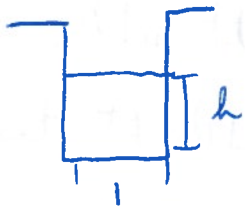


Q is not exactly function of ρ.

Flood velocity is bigger here

$Q = Q(\rho)$ is a decent model, but not very accurate (e.g. floods)

$Q = \rho u = h u$



- Mass conservation: $h_t + (hu)_x = 0$
- Momentum conservation:

variation of momentum is different of pressures:

$$(hu)_t + \left(hu^2 + g\frac{h^2}{2}\right)_x = 0$$

momentum — flux of momentum — Pressure

$h = H + \eta$, $\eta \ll H$, $u \ll 1$

Linearize equations and we get

- Mass conservation : $\eta_t + H u_x = 0$
- Momentum conservation : $H u_t + gH \eta_x = 0$

$$\begin{cases} \eta_t + H u_x = 0 \\ u_t + g\eta_x = 0 \end{cases}$$

$$\Downarrow$$

$$\eta_{tt} - \underbrace{gH}_{c^2} \eta_{xx} = 0$$

Wave equation.

Speed of Tsunami

$$c = \sqrt{gH} \approx \sqrt{10 \times 4000} = 200 \text{ m s}^{-1}$$

H: 3km deep

$$\approx 720 \text{ km h}^{-1}$$

Back to non-linear equations. Developing momentum conservation

$$h u_t + h u u_x + g h h_x = 0 \implies u_t + u u_x + g h_x = 0$$

$$\underbrace{+(hu)_x + h_t}_{=0 \text{ By mass conservation}}$$

we forget about shocks for a while

$$\begin{pmatrix} h \\ u \end{pmatrix}_t + \underbrace{\begin{pmatrix} u & h \\ g & u \end{pmatrix}}_{A} \begin{pmatrix} h \\ u \end{pmatrix}_x = 0$$

In the linear case the matrix is $\begin{pmatrix} 0 & H \\ g & 0 \end{pmatrix}$.

We're going to pick vector $(l_1 \; l_2)$ and consider

$$(l_1 \; l_2) \begin{pmatrix} h \\ u \end{pmatrix}_x + (l_1 \; l_2) A \begin{pmatrix} h \\ u \end{pmatrix}_x = 0$$

We will chose $(l_1 \; l_2)$ as left eigenvector of A.

Nobbling - 3

Complex potential $\Phi(z) = \phi(x,y) + i\psi(x,y)$.

ϕ: velocity potential, ψ: stream function