Representations of Thompson groups and Cuntz algebras

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Motivation

Representations of Cuntz algebra $\mathcal{O}_2$

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Representations of Thompson groups

What we want to study:

- Unitary equivalence of representations
- Irreducible representations
- ...

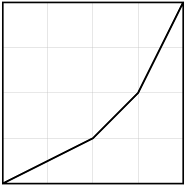
Thompson groups $F \subset T \subset V$ (1965)

Definition (Group F)

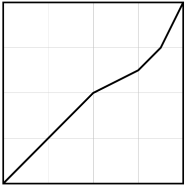
Set of piecewise linear bijections of $[0, 1]$ such that:

- are homeomorphisms of $[0, 1]$;
- have a finite set of points of non-differentiability, which are dyadic numbers;
- derivate is always a power of 2 when it exists;
- maps the dyadics in $[0, 1]$ bijectively onto themselves.

Thompson group F

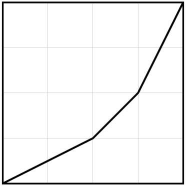


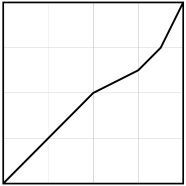
A



B

$$A(x) = \begin{cases} \frac{1}{2}x & \text{for} \quad 0 \le x \le \frac{1}{2} \\ x - \frac{1}{4} & \text{for} \quad \frac{1}{2} \le x \le \frac{3}{4} \\ 2x - 1 & \text{for} \quad \frac{3}{4} \le x \le 1 \end{cases} \qquad B(x) = \begin{cases} x & \text{for} \quad 0 \le x \le \frac{1}{2} \\ \frac{1}{2}x + \frac{1}{4} & \text{for} \quad \frac{1}{2} \le x \le \frac{3}{4} \\ x - \frac{1}{8} & \text{for} \quad \frac{3}{4} \le x \le \frac{7}{8} \\ 2x - 1 & \text{for} \quad \frac{7}{8} \le x \le 1 \end{cases}.$$





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F is generated by A, B and relations

1. $[AB^{-1}, X_2] = 1$
2. $[AB^{-1}, X_3] = 1$

where $[g,h] = ghg^{-1}h^{-1}$, $X_2 = A^{-1}BA$ and $X_3 = A^{-2}BA^2$.

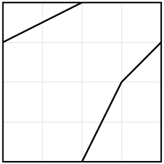
Thompson groups $F \subset T \subset V$ (1965)

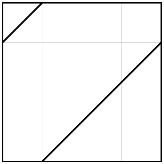
Definition (Group T)

Set of piecewise linear bijections of $[0,1[$ such that:

- are homeomorphisms of $[0,1[$ with the circle topology;
- only have a finite set of points of non-differentiability, which are dyadic numbers;
- derivate is always a power of 2 when it exists;
- maps the dyadics in $[0,1]$ bijectively onto themselves (except 1).

Thompson group T





$$C(x) = \begin{cases} \frac{1}{2}x + \frac{3}{4} & \text{for} \quad 0 \leq x \leq \frac{1}{2} \\ 2x - 1 & \text{for} \quad \frac{1}{2} \leq x \leq \frac{3}{4} \\ x - \frac{1}{4} & \text{for} \quad \frac{3}{4} \leq x < 1 \end{cases} \qquad D(x) = \begin{cases} x + \frac{3}{4} & \text{for} \quad 0 \leq x \leq \frac{1}{4} \\ x - \frac{1}{4} & \text{for} \quad \frac{1}{4} \leq x < 1 \end{cases}.$$

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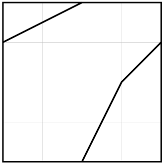

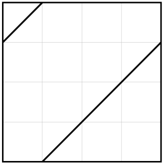

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T is generated by A, B and C and relations

3. $C = BC_2$
4. $C_2X_2 = BC_3$
5. $CA = C_2^2$
6. $C^3 = 1$

where $C_2 = A^{-1}CB$ and $C_3 = A^{-2}CB^2$.

Definition (Group V)

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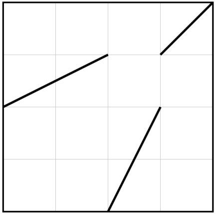
Thompson group V



π_0

$$\pi_0(x) = \begin{cases} \frac{1}{2}x + \frac{1}{2} & \text{for} \quad 0 \leq x < \frac{1}{2} \\ 2x - 1 & \text{for} \quad \frac{1}{2} \leq x < \frac{3}{4} \\ x & \text{for} \quad \frac{3}{4} \leq x < 1 \end{cases}$$

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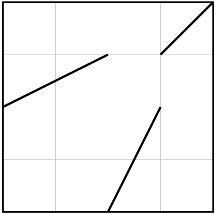

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V is generated by A, B, C and π_0 and relations

7. $\pi_1^2 = 1$
8. $[\pi_1, \pi_3] = 1$
9. $(\pi_2\pi_1)^3 = 1$
10. $[X_3, \pi_1] = 1$
11. $\pi_1 X_2 = B\pi_2\pi_1$
12. $\pi_2 B = B\pi_3$
13. $\pi_1 C_3 = C_3\pi_2$
14. $(\pi_1 C_3)^3 = 1$

where $\pi_1 = C_2^{-1}\pi_0 C_2$, $\pi_2 = A^{-1}\pi_0 A$ and $\pi_3 = A^{-2}\pi_0 B^2$.

C*-algebra

Definition (Banach algebra)

An algebra A over $\mathbb{C}$ is a Banach algebra if it is endowed with a norm that makes it into a Banach space and

$$||ab|| \leq ||a|| \cdot ||b||$$

for all $a, b \in A$.

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We say A is a involutive algebra if there is a map $\cdot^* : A \to A$ such that

1. $(a^*)^* = a$
2. $(ab)^* = b^* a^*$
3. $(\lambda a + b)^* = \overline{\lambda} a^* + b^*$

for all $a, b \in A$ and $\lambda \in \mathbb{C}$.

C*-algebra

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An involutive Banach algebra A is a C*-algebra if it satisfies the relation

$$||aa^*|| = ||a||^2$$

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Examples:

- $\mathbb{C}$;
- $M_{n\times n}(\mathbb{C})$, space of $n \times n$ complex matrices;
- $C(K)$, space of complex-valued functions on a compact K.

C*-algebra

Theorem (Gelfand)

Let A be a commutative unital C*-algebra. Then, there is an isometric *-isomorphism $\phi : A \to C_0(X)$, where X is compact Hausdorff space. If A is only commutative, the A is isomorphic to $C(X)$, for some X locally compact Hausdorff space.

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Theorem (Gelfand-Naimark)

Let A be a C*-algebra. Then A is isometrically *-isomorphic to some C*-subalgebra of bounded operators on a Hilbert space.

Hilbert spaces

Hilbert space: a complete complex linear space with an inner product.

Example

$\ell^2(\mathbb{N}) = \{(a_n) \in \mathbb{C}^{\mathbb{N}} : \sum_{n=1}^{\infty} |a_n|^2 < +\infty\}$, with inner product

$$\langle (a_n), (b_n) \rangle = \sum_{n=1}^{\infty} a_n \overline{b_n}$$

It is true that all separable Hilbert spaces are isometric to $\ell^2(\mathbb{N})$.

Hilbert spaces

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The space of continuous linear operators on a Hilbert space, $B(H)$, forms a Banach algebra with norm

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that satisfies $||T \circ T^*|| = ||T||^2$ (C*-algebra).

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Given $T : H \to H$ we define $T^* : H \to H$ such that

$$\langle Tx, y \rangle = \langle x, T^* y \rangle \quad (\textit{adjoint operator})$$

with $x \in H,\ y \in K$.

Cuntz algebras $\mathcal{O}_n$ (1977)

Definition (Cuntz algebra $\mathcal{O}_n$)

Universal C*-algebra generated by n isometries, $\{s_1, ..., s_n\}$ such that:

- $\sum_{i=1}^{n} s_i s_i^* = I$
- $s_i^* s_j = \delta_{ij} I$ for $i, j \in \{1, ..., n\}$

where I is the identity.

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where I is the identity.

Definition (Cuntz algebra $\mathcal{O}_2$)

Universal C*-algebra generated by two isometries, s_1 and s_2 such that:

- $s_1 s_1^* + s_2 s_2^* = I$
- $s_i^* s_j = \delta_{ij} I$ for $i, j \in \{1, 2\}$

where I is the identity.

Group representations & *-representations

Let H be a Hilbert space.

A unitary group representation of a discrete group G is a group homomorphism $\pi : G \to B(H)$ such that $\pi(g^{-1}) = \pi(g)^*$ for all $g \in G$.

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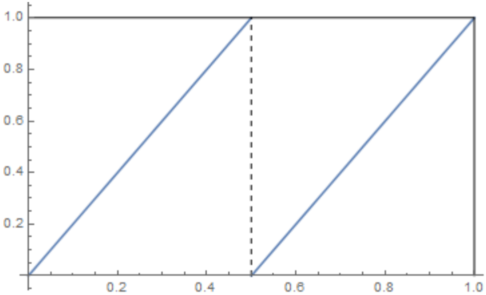
Given two representations of a discrete group G, $\pi_1 : G \to B(H_1)$ e $\pi_2 : G \to B(H_2)$, we say that the representations are unitarily equivalent if there is $U : H_2 \to H_1$ unitary such that

$$\pi_1(g)U = U\pi_2(g)$$

for all $g \in G$. We write $\pi_1 \sim \pi_2$.

$f(x) = 2x \pmod 1$



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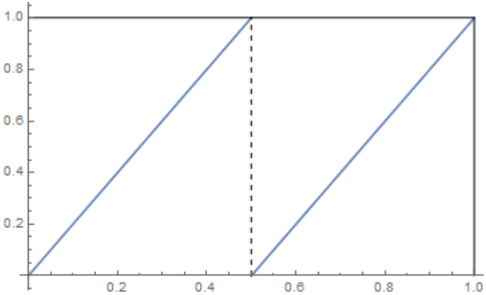

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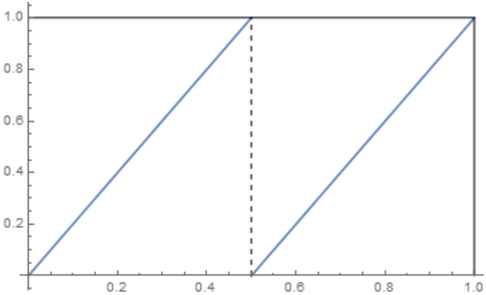

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$$orb(x) = \{f^m(x) : m \in \mathbb{Z}\} \qquad H_x = \ell^2(orb(x))$$

$f(x) = 2x \pmod 1$

Lemma

Let p be a prime. Then

$$orb\left(\frac{1}{p}\right) = \left\{\frac{m}{2^n p} : m \in \mathbb{N},\ n \in \mathbb{N}_0,\ p \nmid m,\ 1 \leq m < 2^n p\right\}.$$

$f(x) = 2x \pmod 1$

H_x has an orthonormal basis $\{\delta_y : y \in \mathrm{orb}(x)\}$ with inner product

$$\langle \delta_y, \delta_z \rangle = \delta_{y,z}$$

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We can define the following operators on H_x:

- $S_1 \delta_y = \delta_{\frac{y}{2}}$
- $S_2 \delta_y = \delta_{\frac{y+1}{2}}$

Their adjoints are given by:

- $S_1^* \delta_y = \delta_{2y}$ for $y \in [0, \frac{1}{2}[$
- $S_2^* \delta_y = \delta_{2y-1}$ for $y \in [\frac{1}{2}, 1[$

Representations of Thompson groups on $B(H_x)$

Lemma (2008, easy proof)

The operators $S_1, S_2 \in B(H_x)$ satisfy the relations of the generators of $\mathcal{O}_2$ and induce a $*$-representation $\pi_x : \mathcal{O}_2 \to B(H_x)$ such that $\pi_x(s_i) = S_i, \;\; i = 1, 2.$

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We now define the following operators on $B(H_x)$:

$$\begin{aligned} A_x &:= S_1 S_1 S_1^* + S_1 S_2 S_1^* S_2^* + S_2 S_2^* S_2^* \\ B_x &:= S_1 S_1^* + S_2 S_1 S_1 S_1^* S_2^* + S_2 S_1 S_2 S_1^* S_2^* S_2^* + S_2 S_2 S_2^* S_2^* S_2^* \\ C_x &:= S_1 S_1^* S_2^* + S_2 S_1 S_2^* S_2^* + S_2 S_2 S_1^* \\ \pi_{0,x} &:= S_1 S_1^* S_2^* + S_2 S_1 S_1^* + S_2 S_2 S_2^* S_2^* \end{aligned}$$

Representations of V on $B(H_x)$

Theorem

The 4 operators defined are unitary and satisfy the 14 relations of V. Thus, we have a group unitary representation $\rho_x : V \to B(H_x)$ of V such that

$$\rho_x(A) = A_x \quad \rho_x(B) = B_x \quad \rho_x(C) = C_x \quad \rho_x(\pi_0) = \pi_{0,x}$$

for $x \in [0, 1]$. Moreover

$$\rho_x(g)\delta_y = \delta_{g(y)}$$

for $g \in F$ and $y \in orb(x)$

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The case $x = \frac{1}{2}$ has already been studied by Kristian Olesen in 2006.

Representations of V on $B(H_x)$

$$\rho_x = \pi_x|_V, \quad \sigma_x = \pi_x|_T, \quad \tau_x = \pi_x|_F$$

Representations of V on $B(H_x)$

$$\rho_x = \pi_x|_V, \quad \sigma_x = \pi_x|_T, \quad \tau_x = \pi_x|_F$$

Question: *What can we say about ρ_x for $x \neq \frac{1}{2}$? And what about τ_x and σ_x?*

What about $x \neq \frac{1}{2}$?

When can we say two representations ρ_x and ρ_y are unitarily equivalent?

$$\rho_x \sim \rho_y \iff ???$$

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$$\rho_x \sim \rho_y \iff \text{???}$$

Theorem (2007)

For the representations π_x of $\mathcal{O}_2$, we have

$$\pi_x \sim \pi_y \iff orb(x) = orb(y)$$

for $x, y \in [0, 1]$.

$\rho_x \sim \rho_y$

When can we say two representations ρ_x and ρ_y are unitarily equivalent?

$$\rho_x \sim \rho_y \iff ???$$

Lemma

Let $x \in [0,1]$. Then $C^*_{\rho_x}(V) = \pi_x(\mathcal{O}_2)$, where $C^*_{\rho_x}(V)$ denotes the C*-algebra generated by $\rho_x(V)$ inside $\pi_x(\mathcal{O}_2)$.

$\rho_x \sim \rho_y$

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Theorem

Let $x, y \in [0,1]$. Then

$$\rho_x \sim \rho_y \iff x \sim y$$

$\tau_x \sim \tau_y$ and $\sigma_x \sim \sigma_y$

What about τ_x and σ_x?
Since $C^*_{\tau_x}(F) \subset C^*_{\sigma_x}(T) \subset C^*_{\rho_x}(V)$, we can't use the same technique.

Theorem

Let $x \in [0, 1]$. Then

$$\tau_x \sim \tau_{\frac{1}{2}} \iff x \sim \frac{1}{2}$$

and

$$\sigma_x \sim \sigma_{\frac{1}{2}} \iff x \sim \frac{1}{2}$$