Topologies with arithmetic properties

Novos Talentos em Matemática

Inês Guimarães

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Ingredients: set X, subset $\mathcal{T} \subseteq 2^X$

Requirements:

- $\emptyset$, $X \in \mathcal{T}$
- $\{A_i\}_{i \in I} \in \mathcal{T} \Rightarrow \bigcup_{i \in I} A_i \in \mathcal{T}$
- $A_1, A_2, \ldots, A_n \in \mathcal{T} \Rightarrow \bigcap_{1 \leq i \leq n} A_i \in \mathcal{T}$

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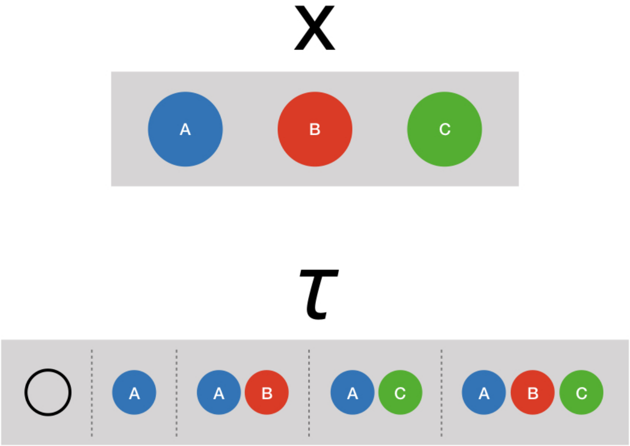
▷ if $\emptyset, X \in \mathcal{B} \subseteq \mathcal{T}$ and $A = \bigcup_{i\in I} B_i$ for $B_i \in \mathcal{B}$ then $\mathcal{B}$ is a basis.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Topology on the integers $\mathbb{Z}$

▷ Basis for $\mathcal{T}_F$

$$\boxed{a\mathbb{Z} + b = \{an + b, n \in \mathbb{Z}\}}$$

for $a \in \mathbb{Z} \setminus \{0\}, b \in \mathbb{Z}$.

E.g. $2\mathbb{Z}$ $\quad 3\mathbb{Z} - 1$ $\quad 42\mathbb{Z} + 6$ $\quad \dots$

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▷ Simultaneously open and closed (**clopen**):

$$a\mathbb{Z} + b = \mathbb{Z} \setminus \bigcup_{i=1}^{a-1} (a\mathbb{Z} + (b + i))$$

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$\{-1, 1\}$ is not open!

Some properties

- metrizable
- totally disconnected
- not compact
- ultraparacompact

Topology on the positive integers $\mathbb{N}$

▷ Basis for $\mathcal{T}_G$

$$a\mathbb{N}_0 + b = \{an + b, n \in \mathbb{Z}_{\geq 0}\}$$

for $a, b \in \mathbb{N}$ and $(a, b) = 1$.

E.g: $2\mathbb{N}_0 + 1 \quad 3\mathbb{N}_0 + 7 \quad 42\mathbb{N}_0 + 25 \quad \ldots$

Some properties

- Hausdorff
- connected
- not regular
- not compact

Dirichlet's Theorem

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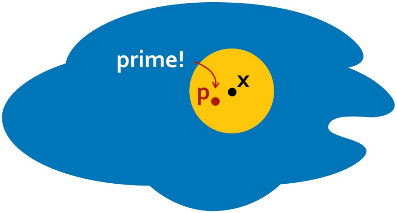
Topological translation:

The set of primes is dense in the set of the positive integers.



$f \in \mathbb{Z}[x]$ non-constant polynomial

$\mathcal{P}_f = \{p \in \mathcal{P} : p \mid f(n) \text{ for some } n \in \mathbb{Z}\}$

Properties:

- Takes on infinitely many composite values
- Is divisible by an infinite amount of primes (that is, $\mathcal{P}_f$ is an infinite set)
- If it is separable, given $p \in \mathcal{P}_f$, there are arbitrarily big numbers $m \in \mathbb{Z}$ such that $p \mid\mid f(m)$

Not so elementary "facts" about integer polynomials

Bunyakovsky conjecture

▷ $f \in \mathbb{Z}[x]$

- positive leading coefficient
- irreducible over the integers
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then $f(n)$ is prime for an infinite amount of positive integers $n \in \mathbb{N}$.

The polynomial $f(x) = x^2 + 1$

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▷ $\pi = a + bi$ is prime iff:

- it's a rational prime ($b = 0$) and $\pi \equiv 3 \pmod 4$ or
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Are there infinitely many Gaussian primes in the set

$$\mathbb{N} + i = \{n + i \mid n \in \mathbb{N}\}?$$

Known generalization of Dirichlet's Theorem:

There is an infinity of primes in $\{\alpha + \beta\delta \mid \delta \in \mathbb{Z}[i]\}$, with $(\alpha, \beta) = 1$.

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- Extending **Fürstenberg's**:

basis $\{\alpha + \beta\delta \mid \delta \in \mathbb{Z}[i] \text{ and } \alpha, \beta \in \mathbb{Z}[i]\}$

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- **Straight Line Closed Set** Topology:

closed sets $\{\alpha + \beta n \mid n \in \mathbb{Z} \text{ and } (\alpha, \beta) = 1\}$

Some possibilities

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None of these work ☹

More reformulations using Fürstenberg's

- $N \in \mathbb{N}$
- $\mathcal{P}_f = \{p \in \mathcal{P} : p = 2 \text{ or } p \equiv 1 \pmod 4\}$
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Another approach

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Prime powers represented by polynomials

- $\mathcal{Z}_p(f) = \{a \in \{0, 1, \ldots, p-1\} \mid f(a) \equiv 0 \pmod{p}\}$
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If we have

$$a + p\mathbb{Z} \not\subseteq \bigcup_{\substack{q \neq p \\ b \in \mathcal{Z}_q(f)}} b + q\mathbb{Z}$$

then $f(a + px) = p^k$ for some $x \in \mathbb{Z}$ and $k \in \mathbb{N}$.

Otherwise, for *almost all* p,

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$$\Downarrow$$

f is reducible

or

$\exists p \in \mathcal{P}_f$ such that $p \mid f(x)$, $\forall x \in \mathbb{Z}$?

Thank you! :)

References

1. H. Fürstenberg, *On the Infinitude of Primes*, Amer. Math. Monthly, **62** (1955), 353.

2. S. W. Golomb, *A Connected Topology for the Integers*, Amer. Math. Monthly, **66**, No. 8, October (1959), 663-666.

3. S. Waldmann, *Topology, An Introduction*, 1st ed. Springer, New York (2014).

4. N. B. Bussey, *Topological Methods in Number Theory: A Discussion of Gaussian Integers and Primes* (Master's Thesis), Memorial University of Newfoundland St. John's, 1974.