Circle actions on Symplectic Manifolds

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What is Symplectic Geometry?

Symplectic Manifold (M, ω)

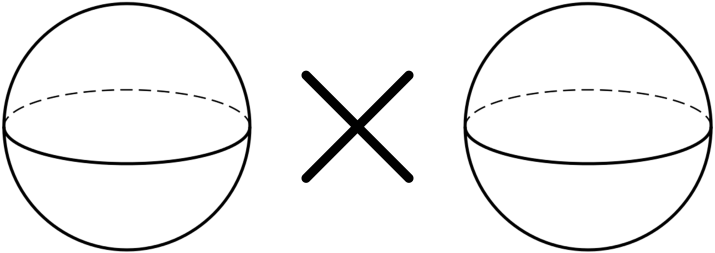
Differential manifold with a 2-form:

- Closed
- Non degenerate

The Manifold

We wish to study a particular manifold (M_{μ,c_1,c_2,c_3,c_4}):

- $(S^2 \times S^2, \omega_\mu \oplus \omega)$ with 4 symplectic blow-ups of sizes c_1, c_2, c_3, c_4. (Assuming $0 < c_4 < c_3 < c_2 < c_1 < c_1 + c_2 < 1$)



Symplectomorphism

Definition

A function $\psi : M \to M$:

- Bijective
- Smooth
- Preserves the symplectic form.

The set of symplectomorphisms is a topological group which we denote by G_{μ,c_1,c_2,c_3,c_4}.

Fundamental Group

- Consider a topological space X and a point $x_0 \in X$. The fundamental group of X based at x_0 is the group of homotopy classes of loops based at x_0.
- Loops that can be continuously deformed into one another are in the same homotopy class.
- A loop can be seen as a continuous function $\phi : S_1 \to X$

Group Action

The action of a group G, on a set X, is a map

$$G \times X \to X$$
$$g, x \mapsto gx$$

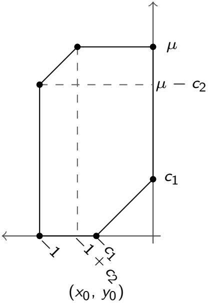
so that $e_G x = x$ and $(gh)x = g(hx)$, $x \in X$, $g, h \in G$.

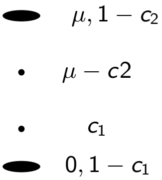
Circle Actions

- $S^1 \times M_{\mu,c_1,c_2,c_3,c_4} \to M_{\mu,c_1,c_2,c_3,c_4}$
  $g, x \mapsto gx$
- If we fix a $g \in S_1$ we get $\varphi_g : M \to M$.
  In the present example φ_g is a symplectomorphism.
- Each circle action induces an element of $\pi_1(G_{\mu,c_1,c_2,c_3,c_4})$.
- $\pi_1(G_{\mu,c_1,c_2,c_3,c_4})$ is generated by S^1 actions.

Polytopes

- Symplectic manifolds of dimension 4,equipped with a toric action ($T^2 = S^1 \times S^1$) are classifed by polytopes in the plane.

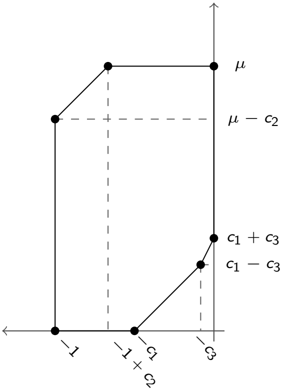




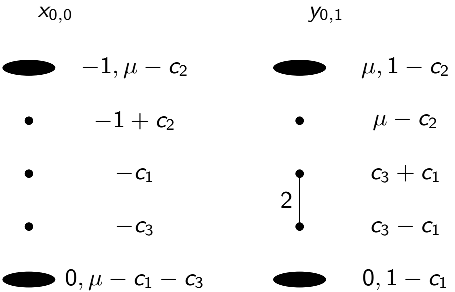
- The projections on the $x - axis$ and $y - axis$ represent the circle actions on the manifold.
- Graphs classify symplectic manifolds equipped with a S^1 action.

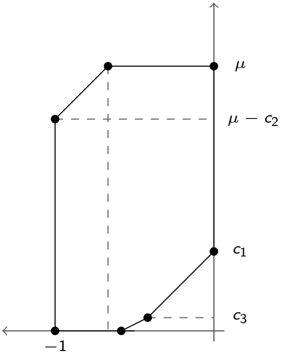
Finding Relations Between the Circle Actions

- Polytopes which differ only for an element of $SL(2,\mathbb{Z})$ represent the same toric manifold.
- Let's apply the matrix $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ in the following examples

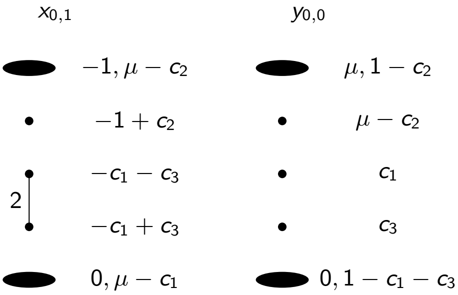

μ
$\mu - c_2$
$c_1 + c_3$
$c_1 - c_3$
-1
$-1 + c_2$
$-c_1$
$-c_3$

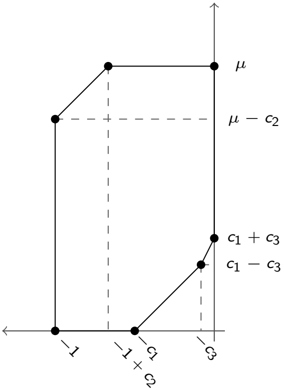
$(x_{0,0}, y_{0,1})$


$x_{0,0}$
$-1, \mu - c_2$
$-1 + c_2$
$-c_1$
$-c_3$
$0, \mu - c_1 - c_3$
$y_{0,1}$
$\mu, 1 - c_2$
$\mu - c_2$
$c_3 + c_1$
2
$c_3 - c_1$
$0, 1 - c_1$

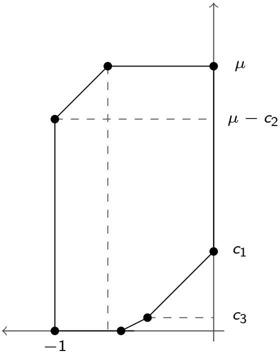

μ
$\mu - c_2$
c_1
c_3
-1

$(x_{0,1}, y_{0,0})$

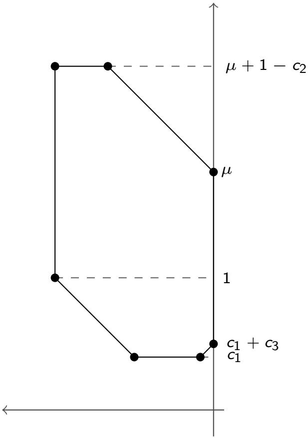

$x_{0,1}$
$y_{0,0}$
$-1, \mu - c_2$
$-1 + c_2$
$-c_1 - c_3$
2
$-c_1 + c_3$
$0, \mu - c_1$
$\mu, 1 - c_2$
$\mu - c_2$
c_1
c_3
$0, 1 - c_1 - c_3$



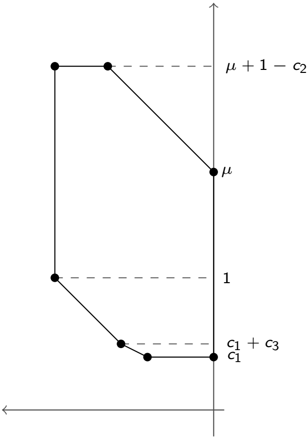
$(x_{0,0}, y_{0,1})$



$(x_{0,1}, y_{0,0})$

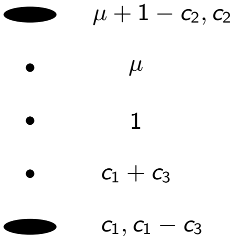


$(x_{0,0}, y_{0,1} - x_{0,0})$

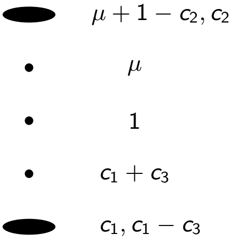


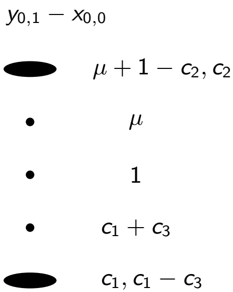
$(x_{0,1}, y_{0,0} - x_{0,1})$

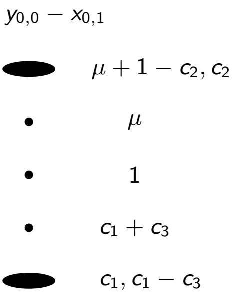
$y_{0,1} - x_{0,0}$



$y_{0,0} - x_{0,1}$



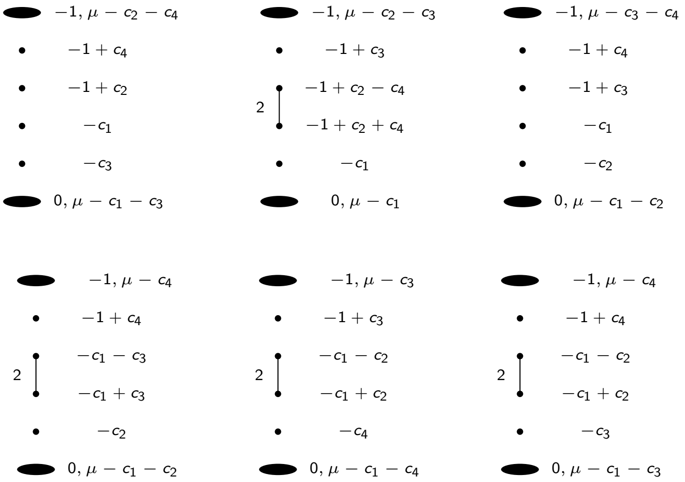




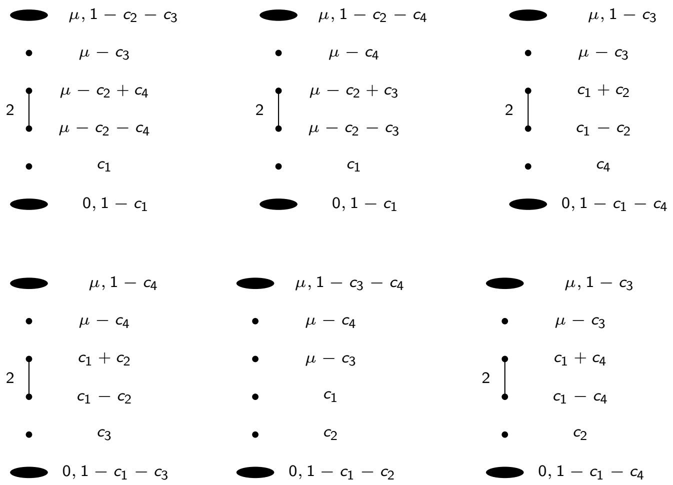
We get: $y_{0,1} - x_{0,0} = y_{0,0} - x_{0,1}$

- In the beginning there were 120 different graphs.
- Only 14 generators.
- Mathematica was a great help!

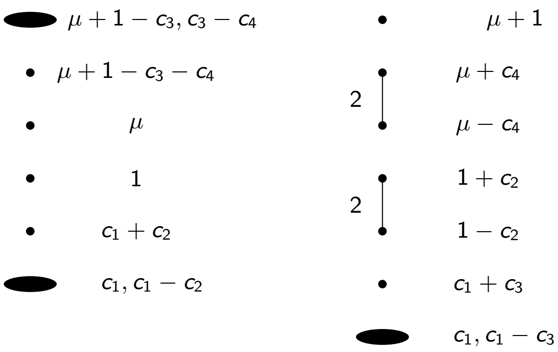
A set of Generators



A set of Generators



A set of Generators



The Conjecture

- The homotopy group of order n ,π_n, is the group of homotopy classes of continuous functions $\phi : S^n \to X$.
- The set of all homotopy groups,π_*, of a topological group is an algebra.
- Conjecture: The full homotopy algebra,$\pi_*(G_{\mu,c_1,c_2,c_3,c_4})$, is generated by elements of the fundamental group (π_1)?

References

- D.McDuff, *What is Symplectic Geometry?* http://math.columbia.edu/∼dusa/
- S.Anjos e S.Eden, *The homotopy Lie algebra of symplectomorphism groups of 3-fold blow-ups of* $(S^2 \times S^2, \sigma_{std} \oplus \sigma_{std})$.to appear in Michigan Math. J. arXiv:1702.03572v2