On Lie modules of Banach space nest algebras

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Banach spaces

A Banach space X is a complete vector space with a norm $\|.\| : X \to \mathbb{R}$, satisfying:

$$\|x\| \geq 0$$
$$\|x\| = 0 \iff x = 0$$
$$\|\alpha x\| = |\alpha|\|x\|$$
$$\|x + y\| \leq \|x\| + \|y\|$$

for all $x, y \in X$, $\alpha \in \mathbb{C}$.

Examples:

- $\mathbb{C}^n$, with the norm $\|(x_1, \ldots, x_n)\| = (\sum_{i=1}^n |x_i|^2)^{1/2}$;
- ℓ^p, $1 \leq p < \infty$, with the norm $\|(x_1, x_2, \ldots)\| = (\sum_{i=1}^\infty |x_i|^p)^{1/p}$;
- $C(K)$, K a compact space, with the norm $\|x\| = \sup_{a \in K} |x(a)|$.

Bounded linear operators

A linear operator $T : X \to Y$ is said to be bounded if there is $c \in \mathbb{R}$ such that

$$\|Tx\| \leq c\|x\| \ \ \forall x \in X.$$

A linear operator is continuous if and only if it is bounded.

The set $\mathscr{B}(X, Y)$ of bounded linear operators $T : X \to Y$ is a Banach space, with norm

$$\|T\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}.$$

The space $\mathscr{B}(X) = \mathscr{B}(X, X)$ is an algebra.
The dual space of X is $X^* = \mathscr{B}(X, \mathbb{C})$.

Nests

A nest $\mathscr{N}$ is a totally ordered set of closed subspaces of X, such that:

- $\{0\}, X \in \mathscr{N}$
- $\wedge\{N_i : i \in I\} = \cap\{N_i : i \in I\} \in \mathscr{N}$
- $\vee\{N_i : i \in I\} = \overline{\mathrm{span}\{N_i : i \in I\}} \in \mathscr{N}$

whenever $\{N_i : i \in I\} \subseteq \mathscr{N}$.

Nests

Example

$X = \mathbb{C}^4$, with a basis $\{e_1, e_2, e_3, e_4\}$.

$\mathscr{N} = \{\{0\},\ \mathrm{span}\{e_1\},\ \mathrm{span}\{e_1, e_2\},\ \mathrm{span}\{e_1, e_2, e_3\},\ \mathbb{C}^4\}$.

Example

$X = C[0,1]$.

$N_s = \{x \in C[0,1] : x(t) = 0 \ \ \forall t \in [s,1]\}$

$\mathscr{N} = \{N_s : s \in [0,1]\} \cup \{C[0,1]\}$.

Nest Algebras

The nest algebra associated with $\mathscr{N}$ is

$$\mathscr{T}(\mathscr{N}) = \{T \in \mathscr{B}(X) : TN \subseteq N \ \ \forall N \in \mathscr{N}\}.$$

$\mathscr{T}(\mathscr{N})$ is a subalgebra of $\mathscr{B}(X)$.

Example

$X = \mathbb{C}^4$.
$\mathscr{N} = \{\{0\}, \ \mathrm{span}\{e_1\}, \ \mathrm{span}\{e_1, e_2\}, \ \mathrm{span}\{e_1, e_2, e_3\}, \ \mathbb{C}^4\}$.
$\mathscr{T}(\mathscr{N})$ is the set of operators represented by upper triangular matrices:

$$\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$$

Modules

A $\mathscr{T}(\mathscr{N})$-bimodule $\mathscr{U}$ is a linear subspace of $\mathscr{B}(X)$ such that

$$\mathscr{U}\mathscr{T}(\mathscr{N}),\ \mathscr{T}(\mathscr{N})\mathscr{U} \subseteq \mathscr{U}.$$

A Lie $\mathscr{T}(\mathscr{N})$-module $\mathscr{L}$ is a linear subspace of $\mathscr{B}(X)$ such that

$$[\mathscr{L}, \mathscr{T}(\mathscr{N})] \subseteq \mathscr{L}.$$

Lie product: $[A, B] = AB - BA$.

Objective

Theorem (L. Oliveira, M. Santos)

Let $\mathscr{N}$ be a nest on a Hilbert space and $\mathscr{L}$ a Lie $\mathscr{T}(\mathscr{N})$-module closed in the weak operator topology (WOT). $\mathscr{T}(\mathscr{N})$-bimodules $\mathscr{J}(\mathscr{L})$ and $\mathscr{K}(\mathscr{L})$ are explicitly constructed such that

$$\mathscr{J}(\mathscr{L}) \subseteq \mathscr{L} \subseteq \mathscr{K}(\mathscr{L}) + \mathscr{D}_{\mathscr{K}(\mathscr{L})},$$

$\mathscr{J}(\mathscr{L})$ is the largest $\mathscr{T}(\mathscr{N})$-bimodule contained in $\mathscr{L}$, $[\mathscr{K}(\mathscr{L}), \mathscr{T}(\mathscr{N})] \subseteq \mathscr{L}$ and $\mathscr{D}_{\mathscr{K}(\mathscr{L})}$ is a subalgebra of the diagonal $\mathscr{D}(\mathscr{N})$.

Bimodules of nest algebras

Theorem

Let $\mathscr{U}$ be a weakly closed $\mathscr{T}(\mathscr{N})$-bimodule. Then there is a left continuous order homomorphism $\phi : \mathscr{N} \to \mathscr{N}$ such that

$$\mathscr{U} = \{T \in \mathscr{B}(X) : TN \subseteq \phi(N) \quad \forall N \in \mathscr{N}\}.$$

Furthermore, $\mathscr{U}$ is the closure (in WOT) of the linear span of its rank one operators.

Bimodules of nest algebras

Example

$X = \mathbb{C}^4$.
$\mathscr{N} = \{\{0\},\ \mathrm{span}\{e_1\},\ \mathrm{span}\{e_1, e_2\},\ \mathrm{span}\{e_1, e_2, e_3\},\ \mathbb{C}^4\}$.

Let $\phi(\{0\}) = \{0\}$,
$\phi(\mathrm{span}\{e_1\}) = \mathrm{span}\{e_1, e_2\}$,
$\phi(\mathrm{span}\{e_1, e_2\}) = \phi(\mathrm{span}\{e_1, e_2, e_3\}) = \phi(\mathbb{C}^4) = \mathrm{span}\{e_1, e_2, e_3\}$.

The $\mathscr{T}(\mathscr{N})$-bimodule $\mathscr{U} = \{T \in \mathscr{B}(X) : TN \subseteq \phi(N) \ \ \forall N \in \mathscr{N}\}$ is the set of matrices of the form

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Bimodules of nest algebras

Every rank one operator $T \in \mathscr{B}(X)$ is of the form $T = f \otimes y$, with $f \in X^*$ and $y \in X$.
Notation: $f \otimes y$ denotes the operator $x \mapsto f(x)y$.

Define

$$N_y = \wedge\{N \in \mathscr{N} : y \in N\}, \quad \hat{N}_f = \vee\{N \in \mathscr{N} : f \in N^\perp\},$$

where $N^\perp = \{g \in X^* : g(N) = \{0\}\}$.

Proposition

Let $\mathscr{U}$ be a weakly closed $\mathscr{T}(\mathscr{N})$-bimodule and $f \otimes y \in \mathscr{U}$. Then

$$g \otimes z \in \mathscr{U}$$

for all $g \in \hat{N}_f^\perp$, $z \in N_y$.

The largest bimodule in $\mathscr{L}$

Theorem

Let $\mathscr{L}$ be a weakly closed Lie $\mathscr{T}(\mathscr{N})$-module. Then the largest $\mathscr{T}(\mathscr{N})$-bimodule contained in $\mathscr{L}$ is

$$\mathscr{J}(\mathscr{L}) = \{T \in \mathscr{B}(X) : TN \subseteq \phi(N) \ \ \forall N \in \mathscr{N}\},$$

where $\phi : \mathscr{N} \to \mathscr{N}$ is given by

$$\phi(N) = \vee\{N_y : \exists f \in X^*,\ f \otimes y \in \mathscr{C}(\mathscr{L}),\ \hat{N}_f < N\}$$

and

$$\mathscr{C}(\mathscr{L}) = \{f \otimes y \in \mathscr{B}(X) : g \otimes z \in \mathscr{L} \ \ \forall g \in \hat{N}_f^{\perp},\ z \in N_y\}.$$

Remark: This theorem is valid for any weakly closed subspace $\mathscr{L}$ of $\mathscr{B}(X)$.

Example

$X = \mathbb{C}^4$.
$\mathscr{N} = \{\{0\},\ \mathrm{span}\{e_1\},\ \mathrm{span}\{e_1, e_2\},\ \mathrm{span}\{e_1, e_2, e_3\},\ \mathbb{C}^4\}$.

Let

$$\mathscr{L} = \left\{ \begin{bmatrix} a & 0 & b & c \\ 0 & a & d & e \\ 0 & 0 & f & g \\ 0 & 0 & h & j \end{bmatrix} : a, b, c, d, e, f, g, h, j \in \mathbb{C} \right\}.$$

$\mathscr{L}$ is a Lie $\mathscr{T}(\mathscr{N})$-module. The largest $\mathscr{T}(\mathscr{N})$-bimodule contained in $\mathscr{L}$ is

$$\mathscr{J}(\mathscr{L}) = \left\{ \begin{bmatrix} 0 & 0 & b & c \\ 0 & 0 & d & e \\ 0 & 0 & f & g \\ 0 & 0 & h & j \end{bmatrix} : b, c, d, e, f, g, h, j \in \mathbb{C} \right\}.$$

Objective

Theorem (L. Oliveira, M. Santos)

Let $\mathscr{N}$ be a nest on a Hilbert space and $\mathscr{L}$ a Lie $\mathscr{T}(\mathscr{N})$-module closed in the weak operator topology (WOT). $\mathscr{T}(\mathscr{N})$-bimodules $\mathscr{J}(\mathscr{L})$ and $\mathscr{K}(\mathscr{L})$ are explicitly constructed such that

$$\mathscr{J}(\mathscr{L}) \subseteq \mathscr{L} \subseteq \mathscr{K}(\mathscr{L}) + \mathscr{D}_{\mathscr{K}(\mathscr{L})},$$

$\mathscr{J}(\mathscr{L})$ is the largest $\mathscr{T}(\mathscr{N})$-bimodule contained in $\mathscr{L}$,
$[\mathscr{K}(\mathscr{L}), \mathscr{T}(\mathscr{N})] \subseteq \mathscr{L}$ and $\mathscr{D}_{\mathscr{K}(\mathscr{L})}$ is a subalgebra of the diagonal $\mathscr{D}(\mathscr{N})$.

Projections

$\mathscr{P}$ – set of bounded linear projections from X to itself, such that:

- For each $N \in \mathscr{N}$, there is a single $P \in \mathscr{P}$ such that $N = \mathrm{ran} P$;
- $PQ = QP$ for all $P, Q \in \mathscr{P}$.

Bimodule $\mathscr{K}(\mathscr{L})$

Define

$$\mathscr{K}(\mathscr{L}) = \mathscr{K}_V + \mathscr{K}_L + \mathscr{K}_D + \mathscr{K}_\Delta,$$

where

$$\begin{aligned}
\mathscr{K}_V &= \overline{\operatorname{span}}^w\{PT(I-P) : P \in \mathscr{P},\ T \in \mathscr{L}\},\\
\mathscr{K}_L &= \overline{\operatorname{span}}^w\{(I-P)TP : P \in \mathscr{P},\ T \in \mathscr{L}\},\\
\mathscr{K}_D &= \overline{\operatorname{span}}^w\{PS(I-P)TP : P \in \mathscr{P},\ T \in \mathscr{L},\ S \in \mathscr{T}(\mathscr{N})\},\\
\mathscr{K}_\Delta &= \overline{\operatorname{span}}^w\{(I-P)TPS(I-P) : P \in \mathscr{P},\ T \in \mathscr{L},\ S \in \mathscr{T}(\mathscr{N})\}.
\end{aligned}$$

Then $\mathscr{K}(\mathscr{L})$ is a weakly closed $\mathscr{T}(\mathscr{N})$-bimodule and $[\mathscr{K}(\mathscr{L}), \mathscr{T}(\mathscr{N})] \subseteq \mathscr{L}$.

Diagonal $\mathscr{D}(\mathscr{N})$

Define

$$\mathscr{D}(\mathscr{N}) = \mathscr{P}' = \{T \in \mathscr{B}(X) : TP = PT \ \ \forall P \in \mathscr{P}\}.$$

Suppose there is a projection $\pi : \mathscr{B}(X) \to \mathscr{D}(\mathscr{N})$ such that

$$\pi(ATB) = A\pi(T)B$$

for all $A, B \in \mathscr{D}(\mathscr{N}),\ T \in \mathscr{B}(X)$.

The largest bimodule in $\mathscr{L}$

Theorem

Let $\mathscr{N}$ be a nest with an associated set $\mathscr{P}$ of commuting projections on a Banach space X, such that there is a projection π of $\mathscr{B}(X)$ onto $\mathscr{D}(\mathscr{N})$ satisfying $\pi(ATB) = A\pi(T)B$ for all $A, B \in \mathscr{D}(\mathscr{N})$, $T \in \mathscr{B}(X)$.
Let $\mathscr{L}$ be a weakly closed Lie $\mathscr{T}(\mathscr{N})$-module. Then there is a weakly closed $\mathscr{T}(\mathscr{N})$-bimodule $\mathscr{K}(\mathscr{L})$ and a subalgebra $\mathscr{D}_{\mathscr{K}(\mathscr{L})}$ of the diagonal $\mathscr{D}(\mathscr{N})$ such that

$$\mathscr{L} \subseteq \mathscr{K}(\mathscr{L}) + \mathscr{D}_{\mathscr{K}(\mathscr{L})}.$$

Furthermore, $[\mathscr{K}(\mathscr{L}), \mathscr{T}(\mathscr{N})] \subseteq \mathscr{L}$.

Some nests satisfying the conditions

- If X has a Schauder basis $\{e_i\}_{i=1}^{\infty}$ and $\mathscr{P}$ is of the form $\mathscr{P} = \{P_n : n \in I\} \cup \{0, I\}$, with

$$P_n\left(\sum_{i=1}^{\infty} \alpha_i e_i\right) = \sum_{i=1}^{n} \alpha_i e_i$$

  and $I \subseteq \mathbb{N}$, then $\mathscr{N}$ satisfies the conditions of the previous theorem.

- If X has finite dimension n, then every nest $\mathscr{N}$ satisfies the conditions, as we can find an appropriate basis $\{e_i\}_{i=1}^{n}$ and a set $\mathscr{P}$ of the above form.

References

- J. A. Erdos and S. C. Power (1982) Weakly closed ideals of nest algebras, *J. Operator Theory* **(2) 7**, 219–235.
- T. D. Hudson, L. W. Marcoux and A. R. Sourour (1998) Lie ideals in triangular operator algebras, *Trans. Amer. Math. Soc.* **(8) 350**, 3321–3339.
- J. Li and F. Lu (2009) Weakly closed Jordan ideals in nest algebras on Banach spaces, *Monatsh. Math.* **156**, 73–83.
- L. Oliveira and M. Santos (2017) Weakly closed Lie modules of nest algebras, *Oper. Matrices* **(1) 11**, 23–35.
- N. K. Spanoudakis (1992) Generalizations of Certain Nest Algebra Results, *Proc. Amer. Math. Soc.* **(3) 115**, 711–723.