Minimal Bimodules of Nest Algebras over Banach Spaces

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Banach Space

Definition

A complex vector space X is a **Banach Space** if:

- it has a norm;
- it is complete.

Nest over a Banach space

Let X be a Banach space.

Let $\mathcal{E}$ be a collection of closed subspaces.

Definition

$\mathcal{E}$ is a **nest** if:

- $\langle 0 \rangle, X \in \mathcal{E}$;
- $\forall N, M \in \mathcal{E}$; $M \subset N$, $N \subset M$ ou $M = N$;
- it is closed under intersections and closed linear spans.

Nest Algebra

Definition

Let $T : X \longrightarrow X$ be a linear operator.

T is **a bounded operator** if $\exists C \in \mathbb{K} : \| Tv \| \leq C \| v \|$.

$\mathcal{B}(X)$ is the set of bounded linear operators.

Nest Algebra

Let X be a Banach space and $\mathcal{E}$ a nest over X.

Definition

We say
$\mathcal{T}(\mathcal{E}) = \{T \in \mathcal{B}(X) : T(N) \subset N; \forall N \in \mathcal{E}\}$
is the **Nest Algebra** of $\mathcal{E}$.

$\mathcal{T}(\mathcal{E})$ is indeed an algebra.

Example

$X = \mathbb{C}^4$

$\mathcal{E} = \{\langle 0 \rangle, \langle e_1 \rangle, \langle e_1, e_2 \rangle, \mathbb{C}^4\}$

$T \in \mathcal{T}(\mathcal{E})$ must be such that $T(E) \subset E, \forall E \in \mathcal{E}$

Example

$X = \mathbb{C}^4$

$\mathcal{E} = \{\langle 0 \rangle, \langle e_1 \rangle, \langle e_1, e_2 \rangle, \mathbb{C}^4\}$

If $T \in T(\mathcal{E})$ it can be represented by a matrix of the form:

$$\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}$$

Example

$X = \ell^p$

$\mathcal{E} = \{\langle 0 \rangle, X\} \cup \{N_n : N_n = \langle e_1, ... e_n \rangle\}$

If $T \in T(\mathcal{E})$ it can be represented by an infinite upper triangular matrix.

Bimodules over $\mathcal{T}(\mathcal{E})$

Let $\mathcal{J} \subset \mathcal{B}(X)$ be a subspace.

Definition

We say $\mathcal{J}$ is a **bimodule** over $\mathcal{T}(\mathcal{E})$ if $\forall J \in \mathcal{J}, \forall T \in \mathcal{T}(\mathcal{E})$:

- $TJ, JT \in \mathcal{J}$

We denote by $\mathcal{J}_0$ the set of **finite rank operators** in $\mathcal{J}$.

Support Functions

Definition

A **support function** is a function $\Phi : \mathcal{E} \longrightarrow \mathcal{E}$ such that:

- $N, M \in \mathcal{E},$ if $M \subset N,$ then $\Phi(M) \subset \Phi(N)$

Example

$X = \mathbb{C}^4$

$\mathcal{E} = \{\langle 0 \rangle, \langle e_1 \rangle, \langle e_1, e_2 \rangle, \langle e_1, e_2, e_3 \rangle, \mathbb{C}^4\}$

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Let $\mathcal{J} = \{T \in \mathcal{B}(X) : T$ is a matrix of the form$\}$

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Let us build a function $\Phi : \mathcal{E} \longrightarrow \mathcal{E}$.

Support Functions

Let $\mathcal{J}$ be a closed bimodule over $\mathcal{T}(\mathcal{E})$, define the associated support function by:

$$\Phi_{\mathcal{J}} : \mathcal{E} \longrightarrow \mathcal{E} \text{ such that } \Phi_{\mathcal{J}}(E) = \overline{\mathsf{Span}_{T\in\mathcal{J}}\, T(E)}$$

Support Functions

Let Φ be a support function.
We define the bimodule generated by Φ:

$$M(\Phi) = \{T \in \mathcal{B}(X) : T(E) \subset \Phi(E), \forall E \in \mathcal{E}\}$$

Then we define:

$$M(\Phi)_0 = \{T \in M(\Phi) : T \text{ has finite rank}\}.$$

My work!

$$\mathcal{J} \longrightarrow \Phi_{\mathcal{J}} \longrightarrow M(\Phi_{\mathcal{J}})$$

My work!

$$\mathcal{J} \longrightarrow \Phi_{\mathcal{J}} \longrightarrow M(\Phi_{\mathcal{J}}) \neq \mathcal{J}.$$

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We wish to find the minimal bimodule.

Note that $\mathcal{J} \subset M(\Phi_{\mathcal{J}})$.

My work!

$$\mathcal{J} \longrightarrow \Phi_{\mathcal{J}} \longrightarrow M(\Phi_{\mathcal{J}}) \neq \mathcal{J}.$$

We wish to find the minimal bimodule.

Note that $\mathcal{J} \subset M(\Phi_{\mathcal{J}})$.

Theorem

Let $\mathcal{J}$ be a norm closed bimodule, then $M(\Phi_{\mathcal{J}})_0 \subset \mathcal{J}$.

Operators of Rank 1

Lemma [L. Duarte, 2018]

Let $T \in M(\Phi_{\mathcal{J}})_0$.
Then, T can be written as a sum of finitely many operators of rank 1.

My work!

If X is a reflexive Banach space or $\mathcal{E}$ is atomic, we can prove:

Lemma

Let $R \in M(\Phi_{\mathcal{J}})$ have rank 1.
Then, $R \in \mathcal{J}$.

My work!

If X is a reflexive Banach space or $\mathcal{E}$ is atomic, we can prove:

Lemma

Let $R \in M(\Phi_{\mathcal{J}})$ have rank 1.
Then, $R \in \mathcal{J}$.

Theorem

$M(\Phi_{\mathcal{J}})_0 \subset \mathcal{J}$.

The main result

Theorem

Let $\mathcal{J}$ be a norm closed bimodule with support function pair (Φ, Ψ). Then,

$$M(\Phi_{\mathcal{J}}, \Psi_{\mathcal{J}}) = M^0(\Psi) + M(\Phi)_0 \subset \mathcal{J}$$

References

- K. R. Davidson, A. P. Donsig and T. D. Hudson, *Norm-closed bimodules of nest algebras*, J. Oper. Theory 39 (1998), 59–87.
- L. Duarte, *Bimodules of Nest Algebras on Banach Spaces*, 2018.