Riemannian Manifolds Dual to Static Spacetimes

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Postulates of Special Relativity

1. The laws of physics are the same for all non-accelerating observers.
2. The speed of light in free space has the same value - c - in all inertial frames.

Space + Time $\rightarrow$ Spacetime

$$(x, y, z) \rightarrow (t, x, y, z)$$

The Relativistic Invariant - Interval ds^2

$$\boxed{ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2}$$

Minkowski Spacetime

What about accelerated motion - gravity ?

Mass and Energy $\leftrightarrow$ Shape of the Spacetime

Curved Spaces - Curvature

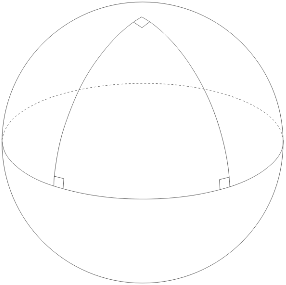


Figure 1: Sum of the internal angles of a triangle $> \pi$

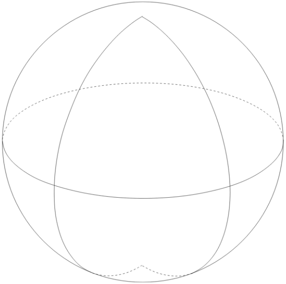
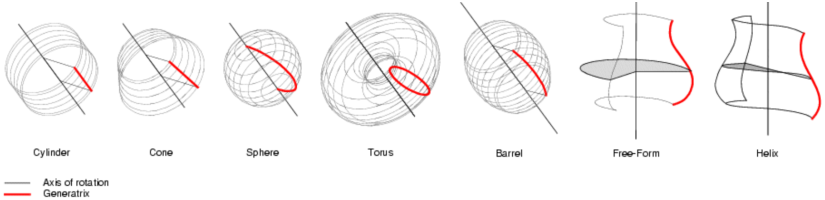


Figure 2: Polygon with two sides

Riemannian Metric

$$\boxed{ds^2 \ = \ g_{ij} dx^i dx^j \ ; \ ds^2 \ > \ 0 \quad}.$$

Geodesics on Surface of Revolution



A possible parametrization of a surface of revolution is:

$$\sigma(z,\phi) = (f(z)\cos\phi, f(z)\sin\phi, z) \quad \rightarrow \boxed{ds^2 = f(z)^2 d\phi^2 + \left[1 + f'(z)^2\right] dz^2}$$

Working conditions:

- Timelike separated events - $ds^2 < 0$
- Static Spacetime - $ds^2 = -e^{2\phi(\vec{x})} dt^2 + g_{ij}(\vec{x}) dx^i dx^j$

Defining $d\tau^2 = -ds^2$, one obtains

$$d\tau^2 = e^{2\phi} dt^2 - g_{ij} dx^i dx^j$$

which yields

$$\boxed{dt^2 = e^{-2\phi} \left(d\tau^2 + g_{ij} dx^i dx^j \right)} \quad \rightarrow \quad \text{Riemannian Metric}$$

Schwarzschild Spacetime

The Spacetime and the corresponding Riemannian Manifold

Lorentzian Metric:

$$\boxed{ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2} \quad *$$

- Vanishing Ricci tensor and scalar curvature.

Riemannian Metric:

$$\boxed{dt^2 = \left(1 - \frac{2m}{r}\right)^{-1} \left[d\tau^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2\right]}$$

- Diagonal non-zero Ricci tensor;
- Scalar curvature, $S = -\frac{12m^2}{r^4}$.

$^* \mathrm{d}\Omega^2 = d\theta^2 + \sin\theta^2 d\phi^2$

Schwarzschild Spacetime

Null geodesics in the equatorial plane, $d\tau = d\theta = 0$

Resulting metric:

$$dt^2 = \left(1 - \frac{2m}{r}\right)^{-1} \left[\left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\phi^2\right]$$

Defining ρ and z so that: $\rho^2 = \left(1 - \frac{2m}{r}\right)^{-1} r^2$ and $d\rho^2 + dz^2 = \left(1 - \frac{2m}{r}\right)^{-2} dr^2$.

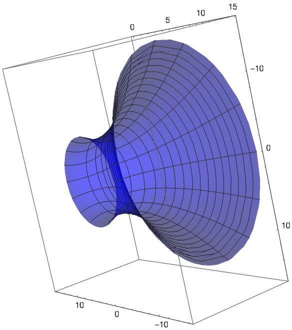


Figure 3: Schwarschild Spacetime ; $d\tau = d\theta = 0$

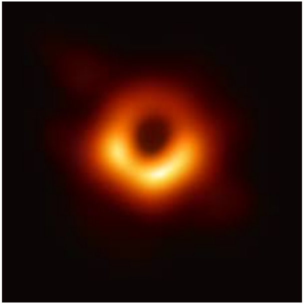


Figure 4: Black Hole

Schwarzschild Spacetime

Radial motion, $d\phi = d\theta = 0$

Resulting metric:

$$dt^2 = \left(1 - \frac{2m}{r}\right)^{-1} \left[d\tau^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2\right]$$

Defining ρ and z such that: $\rho^2 = \left(1 - \frac{2m}{r}\right)^{-1}$ and $d\rho^2 + dz^2 = \left(1 - \frac{2m}{r}\right)^{-2} dr^2$

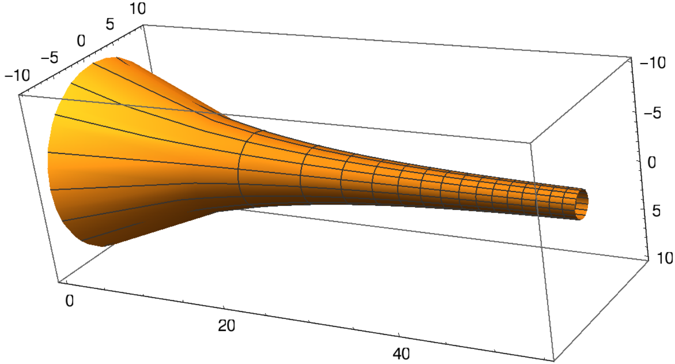


Figure 5: Schwarschild Spacetime ; $d\theta = d\phi = 0$

Lorentzian Metric:

$$ds^2 = -\left[\frac{3}{2}\left(1-\frac{2m}{R}\right)^{\frac{1}{2}} - \frac{1}{2}\left(1-\frac{2m}{R^3}r^2\right)^{\frac{1}{2}}\right]^2 dt^2 + \left(1-\frac{2m}{R^3}r^2\right)dr^2 + r^2 d\Omega^2$$

where R stands for the radius of the spherical body.

Riemannian Metric:

$$dt^2 = \left[\frac{3}{2}\left(1-\frac{2m}{R}\right)^{\frac{1}{2}} - \frac{1}{2}\left(1-\frac{2m}{R^3}r^2\right)^{\frac{1}{2}}\right]^{-2}\left[d\tau^2 + \left(1-\frac{2m}{R^3}r^2\right)dr^2 + r^2 d\Omega^2\right]$$

Schwarzschild Spacetime - Global Solution

Null geodesics in the equatorial plane, $d\tau = d\theta = 0$

Defining ρ and z such that: $\rho^2 = \left[\frac{3}{2}\left(1 - \frac{2m}{R}\right)^{\frac{1}{2}} - \frac{1}{2}\left(1 - \frac{2m}{R^3}r^2\right)^{\frac{1}{2}}\right]^{-2} r^2$ and

$$d\rho^2 + dz^2 = \left[\frac{3}{2}\left(1 - \frac{2m}{R}\right)^{\frac{1}{2}} - \frac{1}{2}\left(1 - \frac{2m}{R^3}r^2\right)^{\frac{1}{2}}\right]^{-2}\left(1 - \frac{2m}{R^3}r^2\right) dr^2.$$

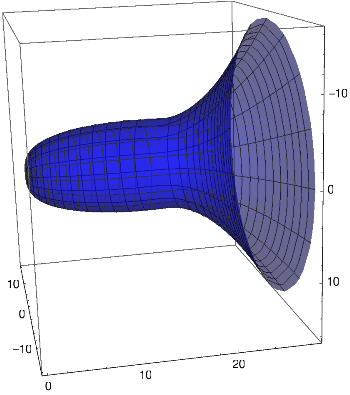


Figure 6: Global Solution ; $d\tau = d\theta = 0$

Schwarzschild Spacetime - Global Solution

Radial motion, $d\phi = d\theta = 0$

Defining $\rho^2 = \left[\frac{3}{2}\left(1 - \frac{2m}{R}\right)^{\frac{1}{2}} - \frac{1}{2}\left(1 - \frac{2m}{R^3}r^2\right)^{\frac{1}{2}}\right]^{-2}$ and

$$d\rho^2 + dz^2 = \left[\frac{3}{2}\left(1 - \frac{2m}{R}\right)^{\frac{1}{2}} - \frac{1}{2}\left(1 - \frac{2m}{R^3}r^2\right)^{\frac{1}{2}}\right]^{-2}\left(1 - \frac{2m}{R^3}r^2\right)dr^2.$$

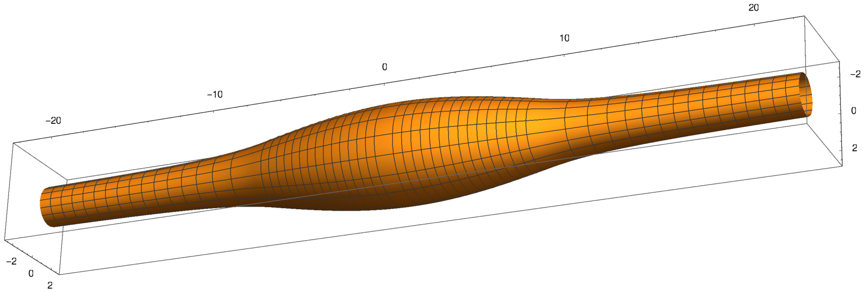


Figure 7: Global Solution ; $d\theta = d\phi = 0$

Lorentzian Metric:

$$ds^2 = -\left(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2\right) dt^2 + \left(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2\right)^{-1} dr^2 + r^2 d\Omega^2$$

where Λ stands for the cosmological constant, $\Lambda > 0$

Riemannian Metric:

$$dt^2 = \left(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2\right)^{-1} \left[d\tau^2 + \left(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2\right)^{-1} dr^2 + r^2 d\Omega^2\right]$$

Schwarzschild de Sitter Spacetime

Radial motion, $d\phi = d\theta = 0$

Resulting metric:

$$dt^2 = \left(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2\right)^{-1}\left[d\tau^2 + \left(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2\right)^{-1}dr^2\right]$$

Defining ρ and z such that: $\rho^2 = \left(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2\right)^{-1}$ and $d\rho^2 + dz^2 = \left(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2\right)^{-2}dr^2$

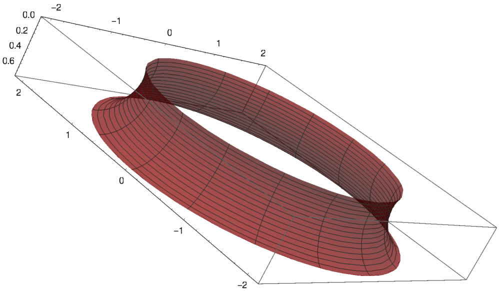


Figure 8: Schwarzschild de Sitter spacetime ; $d\theta = d\phi = 0$

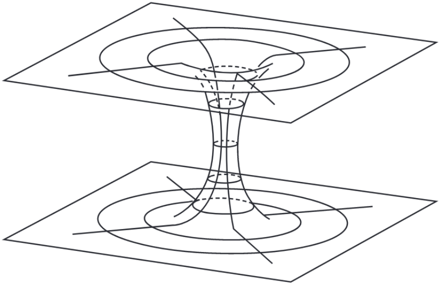
Wormhole

Matching of an interior solution with an exterior Schwarzschild solution

Lorentzian Metric:

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega^2$$

- $\Phi(r)$ describes the gravitational redshift;
- $b(r)$ determines the shape of the wormhole



Choosing:$\Phi(r) = \Phi_0$ and $b(r) = (r_0 r)^{\frac{1}{2}}$:

Wormhole

Radial motion, $d\phi = d\theta = 0$

Defining ρ and z so that: $\rho^2 = \left(1 - \sqrt{\frac{r_0}{a}}\right)^{-1}$ and $d\rho^2 + dz^2 = \rho^2 \left(1 - \sqrt{\frac{r_0}{r}}\right)^{-1} dr^2$, for $r_0 \leq r \leq a$, and $\rho^2 = \left(1 - \frac{\sqrt{r_0 a}}{r}\right)^{-1}$ and $d\rho^2 + dz^2 = \rho^4 dr^2$, for $a \leq r \leq \infty$.

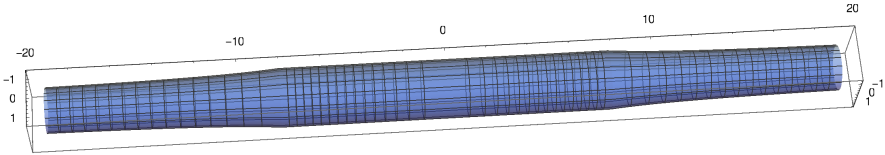


Figure 9: Wormhole ; $d\theta = d\phi = 0$

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