Hydrodynamic limit for weakly asymmetric systems

Pedro Capitão

Instituto Superior Técnico
Universidade de Lisboa

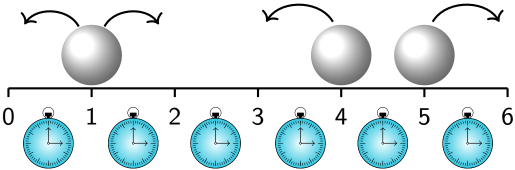
7 September 2019

Interacting particle systems

- Interacting particle systems are generally continuous time Markov processes with a state space of configurations, such as $\{0,1\}^n$ or $\{0,1\}^{\mathbb{Z}^n}$ (where $n \in \mathbb{N}$).
- The process of configurations is Markovian, but the movement of each particle is not.
- These systems can model the time evolution of:
  - a gas or fluid in a volume,
  - the genetic types of a biological population,
  - the spreading of an infection,
  - the spin of atoms in a magnetic material.

Example: symmetric exclusion process

- **Exclusion rule:** at most one particle can occupy each site.
- **Markov property:** the times between jumps are independent exponentially distributed random variables.

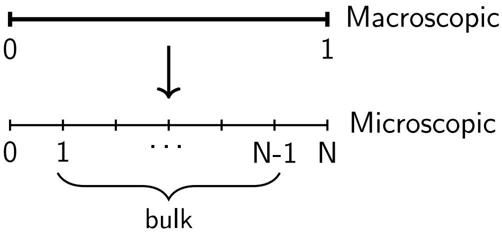


A **configuration** is a function $\eta : \{0, 1, \ldots, 6\} \to \{0, 1\}$, where

$$\eta(x) = \begin{cases} 1, & \text{if site } x \text{ is occupied} \\ 0, & \text{if site } x \text{ is empty} \end{cases} \qquad x = 0, 1, \ldots, 6.$$

The model

Discretize the interval $[0,1]$ into $N \in \mathbb{N}$ identical subintervals:

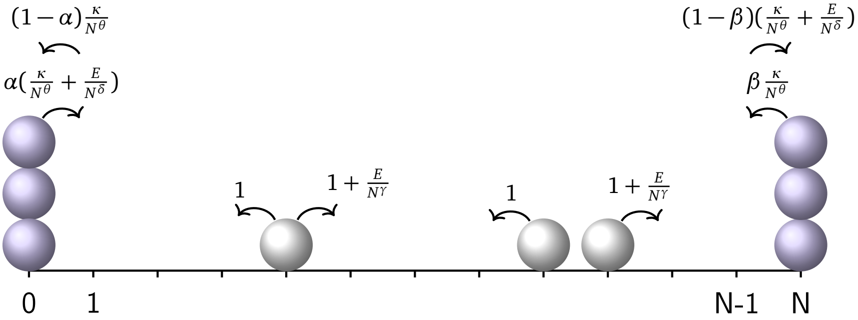


- A site $x \in \{0, 1, \dots, N\}$ in the microscopic scale corresponds to the point $\frac{x}{N} \in [0,1]$ in the macroscopic scale.
- We consider, for each $N \in \mathbb{N}$, a process $\{\eta_t^N : t \geq 0\}$, with state space $\Omega_N = \{0,1\}^{\{1,\dots,N-1\}}$.

The dynamics

Model parameters:

- $\alpha, \beta \in (0,1)$,
- $E, \kappa > 0$,
- $\theta, \gamma, \delta \geq 0$.



The generator

The infinitesimal generator of the Markov process $\{\eta_t^N : t \geq 0\}$, with state space $\Omega_N = \{0,1\}^{\{1,\dots,N-1\}}$, is the operator $\mathscr{L}_N$, which acts on functions $f : \Omega_N \to \mathbb{R}$ as

$$(\mathscr{L}_N f)(\eta) = \sum_{x=0}^{N-1} c_{x,x+1}(\eta)\big(f(\eta^{x,x+1}) - f(\eta)\big),$$

where, for $1 \leq x \leq N-2$,

$$c_{x,x+1}(\eta) = \Big(1 + \frac{E}{N^\gamma}\Big)\eta(x)[1-\eta(x+1)] + \eta(x+1)[1-\eta(x)],$$
$$c_{0,1}(\eta) = \Big(\frac{\kappa}{N^\theta} + \frac{E}{N^\delta}\Big)\alpha[1-\eta(1)] + \frac{\kappa}{N^\theta}\eta(1)[1-\alpha],$$
$$c_{N-1,N}(\eta) = \Big(\frac{\kappa}{N^\theta} + \frac{E}{N^\delta}\Big)\eta(N-1)[1-\beta] + \frac{\kappa}{N^\theta}\beta[1-\eta(N-1)].$$

Hydrodynamic limit

- We want to derive the partial differential equations which describe the space-time evolution of the density of particles in the system.
- In order to observe a non trivial hydrodynamics evolution, we must rescale time by N^2 at the microscopic level: instead of $\{\eta_t^N : t \geq 0\}$, we consider the process $\{\eta_{tN^2}^N : t \geq 0\}$, which has generator $N^2 \mathscr{L}_N$.

For any configuration $\eta \in \Omega_N$, define the empirical measure

$$\pi^N(\eta) = \frac{1}{N-1} \sum_{x=1}^{N-1} \eta(x) \delta_{\frac{x}{N}}.$$

and, for any trajectory $\eta_\cdot \in \mathscr{D}([0,T], \Omega_N)$, let $\pi_t^N(\eta_\cdot) = \pi^N(\eta_{tN^2})$.

Hydrodynamic limit

Notation: $\langle \pi^N(\eta), G\rangle = \int_0^1 G(q)\pi^N(\eta, dq) = \frac{1}{N-1}\sum_{x=1}^{N-1} \eta(x)G(\frac{x}{N})$.

- **Hypothesis:** Let $\rho_0 : [0,1] \to [0,1]$ be a measurable function and let $\{\mu_N\}_{N\in\mathbb{N}}$ be a sequence of probability measures on Ω_N such that, for any continuous function $G : [0,1] \to \mathbb{R}$ and any $\varepsilon > 0$

$$\lim_{N\to\infty} \mu_N\Big(\eta \in \Omega_N : \Big|\langle \pi^N(\eta), G\rangle - \int_0^1 G(q)\rho_0(q)dq\Big| > \varepsilon\Big) = 0.$$

- **Goal:** Show that, for any $t \in [0,T]$, any continuous function $G : [0,1] \to \mathbb{R}$ and any $\varepsilon > 0$

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\Big(\eta_\cdot : \Big|\langle \pi_t^N(\eta_\cdot), G\rangle - \int_0^1 G(q)\rho(t,q)dq\Big| > \varepsilon\Big) = 0,$$

where $\rho(t,q)$ is the unique weak solution of some PDE with initial condition ρ_0 (*hydrodynamic equation*).

Notion of weak solution

We say that $\rho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the viscous Burgers equation with Dirichlet boundary conditions

$$\begin{cases} \partial_t \rho = \partial_q^2 \rho - E\, \partial_q \sigma(\rho)\,, \\ \rho(t,0) = a\,, \quad \rho(t,1) = b\,, \quad t \in [0,T], \\ \rho(0,q) = \rho_0(q)\,, \quad q \in [0,1]\,, \end{cases}$$

where $\sigma(\rho) = \rho(1-\rho)$, if $\rho \in L^2([0,T], \mathscr{H}^1)$ and ρ satisfies the weak formulation

$$\int_0^1 \rho_t(q) G_t(q)\, dq - \int_0^1 \rho_0(q) G_0(q)\, dq - \int_0^t \int_0^1 \rho_s(q) \Big(\partial_q^2 + \partial_s \Big) G_s(q)\, dq\, ds$$
$$+ \int_0^t \int_0^1 E \sigma(\rho_s(q)) \partial_q G_s(q) dq ds + \int_0^t \Big(b \partial_q G_s(1) - a \partial_q G_s(0) \Big)\, ds = 0,$$

for all $t \in [0,T]$ and any function $G \in C_0^{1,2}([0,T] \times [0,1])$.

Dynkin's Martingale

Fix any function $G : [0, T] \times [0, 1] \to \mathbb{R}$. By Dynkin's formula,

$$M_t^N(G) = \langle \pi_t^N, G_t \rangle - \langle \pi_0^N, G_0 \rangle - \int_0^t N^2 \mathscr{L}_N \langle \pi_s^N, G_s \rangle \, ds$$

is a martingale (with respect to the filtration $\{\mathscr{F}_t\}_{t \geq 0}$ induced by the process: $\mathscr{F}_t = \sigma(\{\eta_s : s < t\})$).
Since the expectation of a martingale is constant and $M_0^N(G) = 0$, we have, for any $t \in [0, T]$,

$$0 = \mathbb{E}_{\mu_N}\Big[\langle \pi_t^N, G_t \rangle - \langle \pi_0^N, G_0 \rangle - \int_0^t N^2 \mathscr{L}_N \langle \pi_s^N, G_s \rangle \, ds \Big].$$

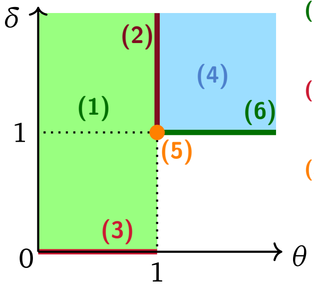
This equation allows us to find out what PDE's we will obtain.

Limit PDE's

If $\gamma = 1$, the density $\rho(t,q)$ is a weak solution of the viscous Burgers equation:

$$\partial_t \rho = \partial_q^2 \rho - E\,\partial_q \sigma(\rho),$$

where $\sigma(\rho) = \rho(1-\rho)$, with the following boundary conditions:



(1) $\begin{cases} \rho(t,0) = \alpha \\ \rho(t,1) = \beta \end{cases}$ (2) $\begin{cases} \partial_q \rho(t,0) = \kappa(\rho_t(0) - \alpha) - E\sigma(\rho_t(0)) \\ \partial_q \rho(t,1) = \kappa(\beta - \rho_t(1)) - E\sigma(\rho_t(1)) \end{cases}$

(3) $\begin{cases} \rho(t,0) = \alpha + E\sigma(\alpha) \\ \rho(t,1) = \beta + E\sigma(\beta) \end{cases}$ (4) $\begin{cases} \partial_q \rho(t,0) = -E\sigma(\rho_t(0)) \\ \partial_q \rho(t,1) = -E\sigma(\rho_t(1)) \end{cases}$

(5) $\begin{cases} \partial_q \rho(t,0) = (\kappa + E\alpha)\rho_t(0) - (\kappa + E)\alpha - E\sigma(\rho_t(0)) \\ \partial_q \rho(t,1) = -(\kappa + E(1-\beta))\rho_t(1) + \kappa\beta - E\sigma(\rho_t(1)) \end{cases}$

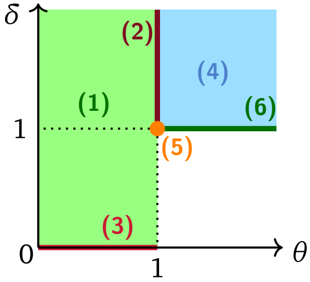
(6) $\begin{cases} \partial_q \rho(t,0) = -E\alpha(1-\rho_t(0)) - E\sigma(\rho_t(0)) \\ \partial_q \rho(t,1) = -E(1-\beta)\rho_t(1) - E\sigma(\rho_t(1)) \end{cases}$

Limit PDE's

If $\gamma > 1$, the density $\rho(t,q)$ is a weak solution of the heat equation:

$$\partial_t \rho = \partial_q^2 \rho$$

with the following boundary conditions:



$$(1)\begin{cases}\rho(t,0)=\alpha\\ \rho(t,1)=\beta\end{cases} \qquad (2)\begin{cases}\partial_q\rho(t,0)=\kappa(\rho_t(0)-\alpha)\\ \partial_q\rho(t,1)=\kappa(\beta-\rho_t(1))\end{cases}$$

$$(3)\begin{cases}\rho(t,0)=\alpha+E\sigma(\alpha)\\ \rho(t,1)=\beta+E\sigma(\beta)\end{cases} \qquad (4)\begin{cases}\partial_q\rho(t,0)=0\\ \partial_q\rho(t,1)=0\end{cases}$$

$$(5)\begin{cases}\partial_q\rho(t,0)=(\kappa+E\alpha)\rho_t(0)-(\kappa+E)\alpha\\ \partial_q\rho(t,1)=-(\kappa+E(1-\beta))\rho_t(1)+\kappa\beta\end{cases}$$

$$(6)\begin{cases}\partial_q\rho(t,0)=-E\alpha(1-\rho_t(0))\\ \partial_q\rho(t,1)=-E(1-\beta)\rho_t(1)\end{cases}$$

Hydrodynamic limit

To prove the results, we need to:

- Show that the sequence $\{\mathbb{Q}_N\}_{N\in\mathbb{N}}$ is **tight**, where $\mathbb{Q}_N$ is the probability measure on $\mathscr{D}([0,T],\mathscr{M}^+)$ induced by the Markov process $\{\pi_t^N : t\geq 0\}$ and by $\mathbb{P}_{\mu_N}$.
  - Tightness implies that every subsequence of $\{\mathbb{Q}_N\}_{N\in\mathbb{N}}$ has a further subsequence which is weakly convergent.
- Show that every weak limit point $\mathbb{Q}$ of a subsequence of $\{\mathbb{Q}_N\}_{N\in\mathbb{N}}$ is concentrated on trajectories of measures which are absolutely continuous and whose density is the unique weak solution of the corresponding PDE:

$$\mathbb{Q}\Big(\pi_\cdot : \pi_t(dq) = \rho(t,q)dq \ \text{ and } \rho \text{ is solution of the PDE}\Big) = 1.$$

References

- Patrícia Gonçalves, *Hydrodynamics for symmetric exclusion in contact with reservoirs* (2017).
- Claude Kipnis, Claudio Landim, *Scaling Limits of Interacting Particle Systems*, Springer-Verlag (1999).
- Thomas Liggett, *Interacting Particles Systems*, Springer (1985).
- Thomas M. Liggett, *Continuous Time Markov Processes: an Introduction*, American Mathematical Society (2010).