Paths on random rows of dice

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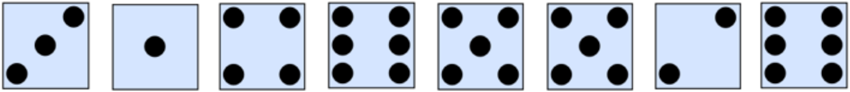
Programa Novos Talentos em Matemática

The Game

The Game



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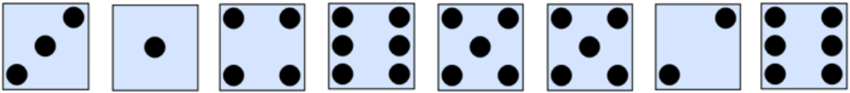

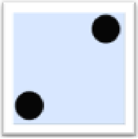

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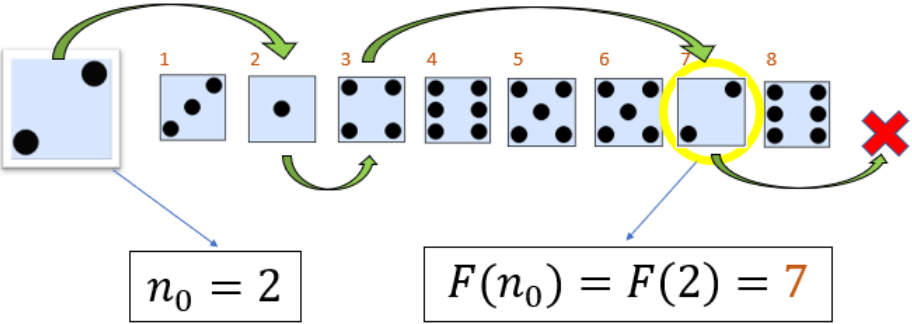

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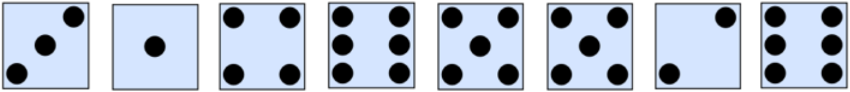

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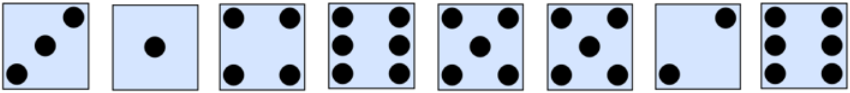

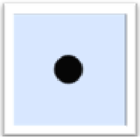

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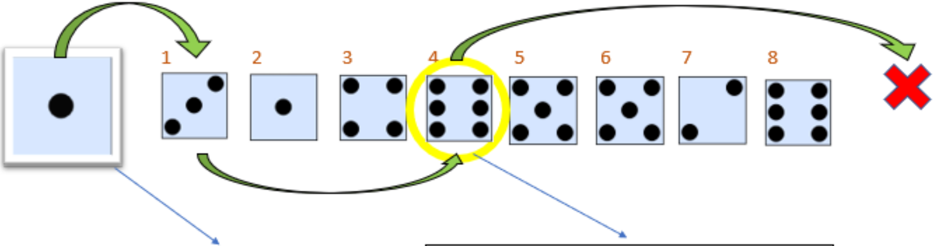


$$n_0 = 1$$

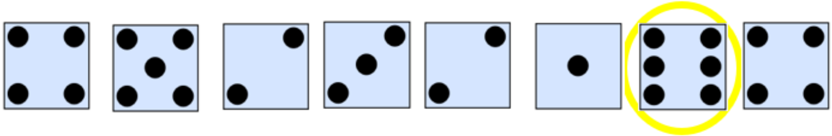
$$F(n_0) = F(1) = 4$$

Good Rows

Good Rows

Good rows are those which satisfy the property

$$F(1) = F(2) = F(3) = F(4) = F(5) = F(6) \quad (1)$$



Questions

Are good rows common?

How many good rows of size N are there?

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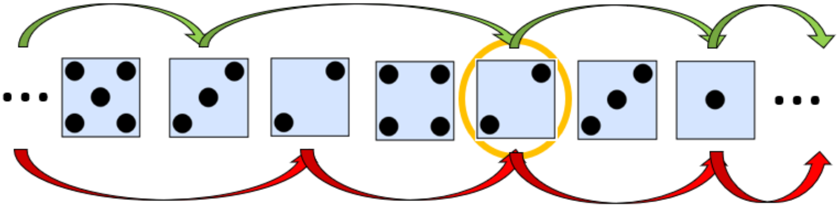

$$\text{Does } \frac{\#\text{good rows of size } N}{\text{total rows of size } N} \longrightarrow 1?$$

Yes!

Convergence

Does $\frac{\#\text{good rows of size } N}{\text{total rows of size } N} \longrightarrow 1$?

Yes!

Small N

$$N = 6 \quad \rightarrow$$

$$N = 7 \quad \rightarrow$$

$$N = 8 \quad \rightarrow$$

$$N = 9 \quad \rightarrow$$

Small N

$$N = 6 \quad \rightarrow \quad 720$$

$$N = 7 \quad \rightarrow$$

$$N = 8 \quad \rightarrow$$

$$N = 9 \quad \rightarrow$$

Small N

$$N = 6 \quad \rightarrow \quad 720$$
$$N = 7 \quad \rightarrow \quad 7920$$
$$N = 8 \quad \rightarrow$$
$$N = 9 \quad \rightarrow$$

Small N

$$N = 6 \quad \to \quad 720$$

$$N = 7 \quad \to \quad 7920$$

$$N = 8 \quad \to \quad 82800$$

$$N = 9 \quad \to$$

Small N

$$
\begin{aligned}
N = 6 \quad &\rightarrow \quad 720 \\
N = 7 \quad &\rightarrow \quad 7920 \\
N = 8 \quad &\rightarrow \quad 82800 \\
N = 9 \quad &\rightarrow \quad 808560
\end{aligned}
$$

Small N

$$
\begin{aligned}
N = 6 \quad &\to \quad 720 \sim 2\% \\
N = 7 \quad &\to \quad 7920 \\
N = 8 \quad &\to \quad 82800 \\
N = 9 \quad &\to \quad 808560
\end{aligned}
$$

Small N

$$N = 6 \quad \rightarrow \quad 720 \sim 2\%$$
$$N = 7 \quad \rightarrow \quad 7920 \sim 3\%$$
$$N = 8 \quad \rightarrow \quad 82800$$
$$N = 9 \quad \rightarrow \quad 808560$$

Small N

$$N = 6 \quad \rightarrow \quad 720 \sim 2\%$$
$$N = 7 \quad \rightarrow \quad 7920 \sim 3\%$$
$$N = 8 \quad \rightarrow \quad 82800 \sim 5\%$$
$$N = 9 \quad \rightarrow \quad 808560$$

Small N

$$N = 6 \quad \rightarrow \quad 720 \sim 2\%$$
$$N = 7 \quad \rightarrow \quad 7920 \sim 3\%$$
$$N = 8 \quad \rightarrow \quad 82800 \sim 5\%$$
$$N = 9 \quad \rightarrow \quad 808560 \sim 8\%$$

Recursion

Let d_1 be the number on the first dice. Then $F(1) = F(1 + d_1)$, so

$$(1)$$

$$\Leftrightarrow$$

$$F(1 + d_1) = F(2) = F(3) = F(4) = F(5) = F(6) \qquad (2)$$

Recursion

$\underline{d_1 = 6:}$

$$(2) \Leftrightarrow F(2) = F(3) = F(4) = F(5) = F(6) = F(7)$$

$\underline{d_1 \in \{1, 2, 3, 4, 5\}:}$

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Recursion

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$\underline{d_1 \in \{1, 2, 3, 4, 5\} :}$

$$(2) \Leftrightarrow F(2) = F(3) = F(4) = F(5) = F(6)$$

Notation

Good rows of size N:

$$[1, 2, 3, 4, 5, 6]_N$$

Rows of size 73 such that $F(1) = F(4) = F(5)$:

$$[1, 4, 5]_{73}$$

Notation

Good rows of size N:

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Rows of size 73 such that $F(1) = F(4) = F(5)$:

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Notation

$$[1, 2, 3, 4, 5, 6]$$

$$[1, 4, 5]$$

Notation

$$[1, 2, 3, 4, 5, 6] \to [* * * * **]$$

$$[1, 4, 5] \to [*// * */]$$

Notation

$$[1, 2, 3, 4, 5, 6] \to [* * * * **] \to [6]$$

$$[1, 4, 5] \to [*// * */] \to [1//2/]$$

Notation

$$[1, 2, 3, 4, 5, 6] \to [* * * * **] \to [6]$$

$$[1, 4, 5] \to [*// * */] \to [1//2/] \to [1//2]$$

Notation

Good rows of size N:

$$[6]_N$$

Rows of size 73 such that $F(1) = F(4) = F(5)$:

$$[1//2]_{73}$$

Back to the recursion

$$[6]_N$$

$d_1 = 6 \to F(2) = F(3) = F(4) = F(5) = F(6) = F(7)$

$1 \quad \times \quad [6]_{N-1}$ possibilities

$d_1 \in \{1, 2, 3, 4, 5\} \to F(2) = F(3) = F(4) = F(5) = F(6)$

$5 \quad \times \quad [5]_{N-1}$ possibilities

$$[6]_N$$

$$d_1 = 6 \to F(2) = F(3) = F(4) = F(5) = F(6) = F(7)$$

1 $\times$ $[6]_{N-1}$ possibilities

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$d_1 \in \{1, 2, 3, 4, 5\} \to F(2) = F(3) = F(4) = F(5) = F(6)$

5 $\times$ $[5]_{N-1}$ possibilities

Back to the recursion

$$[6]_N = [6]_{N-1} + 5 \cdot [5]_{N-1}$$

Back to the recursion

$$[5]_N$$

$d_1 = 5 \to F(2) = F(3) = F(4) = F(5) = F(6)$

$1 \quad \times \quad [5]_{N-1}$ possibilities

$d_1 \in \{1, 2, 3, 4\} \to F(2) = F(3) = F(4) = F(5)$

$4 \quad \times \quad [4]_{N-1}$ possibilities

$d_1 = 6 \to F(2) = F(3) = F(4) = F(5) = F(7)$

$1 \quad \times \quad [4/1]_{N-1}$ possibilities

Back to the recursion

$$[5]_N$$

$d_1 = 5 \to F(2) = F(3) = F(4) = F(5) = F(6)$

$1 \quad \times \quad [5]_{N-1}$ possibilities

$d_1 \in \{1,2,3,4\} \to F(2) = F(3) = F(4) = F(5)$

$4 \quad \times \quad [4]_{N-1}$ possibilities

$d_1 = 6 \to F(2) = F(3) = F(4) = F(5) = F(7)$

$1 \quad \times \quad [4/1]_{N-1}$ possibilities

Back to the recursion

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1 $\times$ $[5]_{N-1}$ possibilities

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4 $\times$ $[4]_{N-1}$ possibilities

$d_1 = 6 \rightarrow F(2) = F(3) = F(4) = F(5) = F(7)$

1 $\times$ $[4/1]_{N-1}$ possibilities

Back to the recursion

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$d_1 = 5 \to F(2) = F(3) = F(4) = F(5) = F(6)$

$1 \times [5]_{N-1}$ possibilities

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$4 \times [4]_{N-1}$ possibilities

$d_1 = 6 \to F(2) = F(3) = F(4) = F(5) = F(7)$

$1 \times [4/1]_{N-1}$ possibilities

Back to the recursion

$$[6]_N = [6]_{N-1} + 5 \cdot [5]_{N-1}$$
$$[5]_N = [5]_{N-1} + 4 \cdot [4]_{N-1} + [4/1]_{N-1}$$

Back to the recursion

$$[6]_N = [6]_{N-1} + 5 \cdot [5]_{N-1}$$
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$$[4]_N = [4]_{N-1} + 3 \cdot [3]_{N-1} + [3/1]_{N-1} + [3//1]_{N-1}$$

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Back to the recursion

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$$[4/1]_N = [5]_{N-1} + 4 \cdot [3/1]_{N-1} + [3/2]_{N-1}$$
$$[3]_N = [3]_{N-1} + 2 \cdot [2]_{N-1} + [2/1]_{N-1} + [2//1]_{N-1} + [2///1]_{N-1}$$
$$[3/1]_N = [4]_{N-1} + 3 \cdot [2/1]_{N-1} + [2/2]_{N-1} + [2/1/1]_{N-1}$$
$$[3//1]_N = [3/1]_{N-1} + 3 \cdot [2//1]_{N-1} + [2/2]_{N-1} + [2//2]_{N-1}$$
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Back to the recursion

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$$[4/1]_N = [5]_{N-1} + 4 \cdot [3/1]_{N-1} + [3/2]_{N-1}$$

$$[3]_N = [3]_{N-1} + 2 \cdot [2]_{N-1} + [2/1]_{N-1} + [2//1]_{N-1} + [2///1]_{N-1}$$

$$[3/1]_N = [4]_{N-1} + 3 \cdot [2/1]_{N-1} + [2/2]_{N-1} + [2/1/1]_{N-1}$$

$$[3//1]_N = [3/1]_{N-1} + 3 \cdot [2//1]_{N-1} + [2/2]_{N-1} + [2//2]_{N-1}$$

$$[3/2]_N = [5]_{N-1} + 4 \cdot [2/2]_{N-1} + [2/3]_{N-1}$$

$$[2]_N = [2]_{N-1} + [1]_{N-1} + [1/1]_{N-1} + [1//1]_{N-1} + [1///1]_{N-1} + [1////1$$

$$[2/1]_N = [3]_{N-1} + 2 \cdot [1/1]_{N-1} + [1/2]_{N-1} + [1/1/1]_{N-1} + [1/1//1]_{N-1}$$

$$[2//1]_N = [2/1]_{N-1} + 2 \cdot [1//1]_{N-1} + [1/2]_{N-1} + [1//2]_{N-1} + [1//1/1]_N$$

Degenerate Cases

$$[2]_N =$$

$$[2]_{N-1} + [1]_{N-1} + [1/1]_{N-1} + [1//1]_{N-1} + [1///1]_{N-1} + [1////1]_{N-1}$$

Degenerate Cases

$$[2]_N =$$

$$[2]_{N-1} + {\color{red}[1]}_{N-1} + [1/1]_{N-1} + [1//1]_{N-1} + [1///1]_{N-1} + [1////1]_{N-1}$$

Degenerate Cases

$$[1]_N = 6^N$$

$$[1]_{N-1} = 6^{N-1}$$

$$[1]_N = 6 \cdot [1]_{N-1}$$

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Degenerate Cases

$$[6]_N = [6]_{N-1} + 5 \cdot [5]_{N-1}$$

Degenerate Cases

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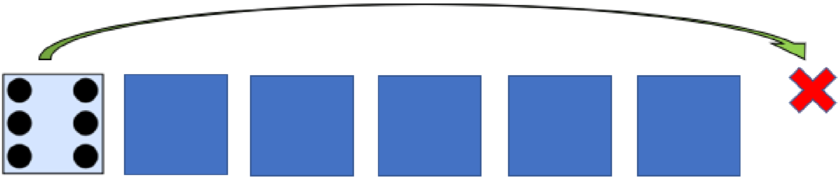
Degenerate Cases

$$[6]_6 = [6]_{\color{red}5} + 5 \cdot [5]_5$$

Degenerate Cases

$$[6]_6 = [6]_5 + 5 \cdot [5]_5$$



Generator of solutions

$$v_n = [[6]_n, [5]_n, \ldots, [1]_n]^T$$

$$v_1 = [0, 0, \ldots, 6]^T$$

$L \rightarrow 63 \times 63$ matrix

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Generator of solutions

$$v_n = L \cdot v_{n-1}, \quad \forall n \geq 2$$

$$v_n = L^{n-1} \cdot v_1 = L^{n-1} \cdot [0, 0, \ldots, 6]^T$$

Generator of solutions

$$v_n = L \cdot v_{n-1}, \quad \forall n \geq 2$$

$$v_n = L^{n-1} \cdot v_1 = L^{n-1} \cdot [0, 0, \ldots, 6]^T$$

Some Results

wxMaxima computes $[6]_N$ in a few seconds, whenever $N \leq 2000$.

$N = 56 \to 37206710775611630508413053666357449744052080$
good rows,
$\sim 98,66\%$ of all rows.

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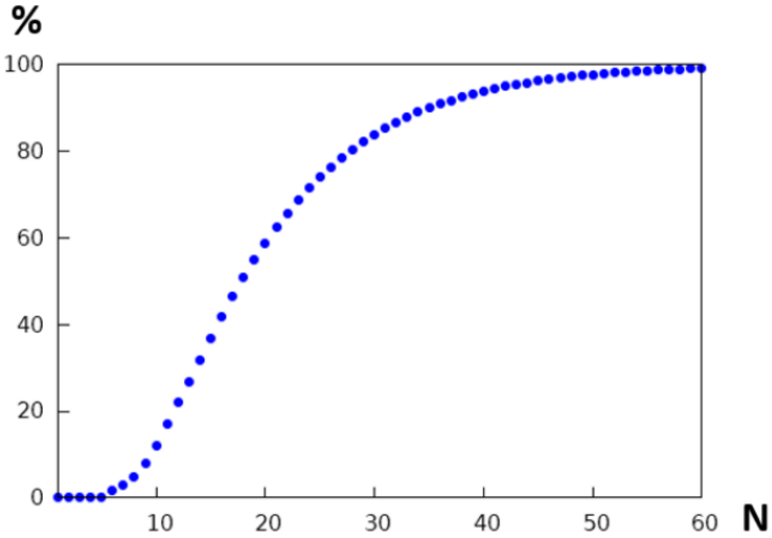

$k \neq 6$

$k = 2$:

Good rows are those such that $F(1) = F(2)$.

The number of bad rows does not depend on N.

There are $2^N - 2$ good rows of size N.

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$\underline{k = 3:}$

$$[3]_N = 3_{N-1} + 2 \cdot [2]_{N-1}$$

$$[2]_N = [2]_{N-1} + [1/1]_{N-1} + [1]_{N-1}$$

$$[1/1]_N = [2]_{N-1} + [/2]_{N-1} + [/1]_{N-1}$$

$$[/2]_N = 3 \cdot [2]_{N-1}$$

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$$[/2]_N = 3 \cdot [2]_{N-1}$$

$$[/1]_N = 3 \cdot [1]_{N-1}$$

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$k = 3$

$$L = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

$k = 3$

$$L = P^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \zeta & 0 \\ 0 & 0 & 0 & 0 & \overline{\zeta} \end{bmatrix} P$$

$k = 3$

There are

$$c_1 + c_2 \cdot 3^N + c_3\lambda^N + c_4\zeta^N + c_5\overline{\zeta}^N$$

good rows of size N where $c_1, c_2, c_3, c_4, c_5, \lambda, \zeta \in \mathbb{C}$ are constants.

$$\frac{[3]_N}{3^N} \longrightarrow 1$$

$$\frac{c_1 + c_2 \cdot 3^N + c_3 \lambda^N + c_4 \zeta^N + c_5 \bar{\zeta}^N}{3^N} \longrightarrow 1$$

$$c_2 = 1$$

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