Embeddings of Cuntz Algebras and applications to representations of Higman-Thompson groups

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Definition

A linear space V with a inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that all Cauchy sequences in V converge to an element of V is denoted a Hilbert space.

Examples:

- $\mathbb{R}^n$ is a Hilbert space with the usual dot product.
- $\ell^2(X)$, the space of functions $f : X \to \mathbb{C}$ such that $\sum_{x \in X} |f(x)|^2 < \infty$ is a Hilbert space.
- $L^2[0,1]$, the space of Lebesgue integrable functions $f : [0,1] \to \mathbb{R}$ such that $\int_0^1 |f(x)|^2 dx < \infty$ with the inner product:

$$\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)} dx.$$

Bounded linear operators

Definition

A function A from a Hilbert space V to a Hilbert space H is said to be a bounded linear operator if it is linear:

$$A(x+y) = A(x) + A(y) \qquad A(\alpha x) = \alpha A(x)$$

and there exists a number $\|A\| \in \mathbb{R}$ such that

$$\|A(x)\| \leq \|A\|\|x\|$$

for any $x \in V$. The space of all bounded linear operators from an Hilbert space H to itself is denoted by $B(H)$.

Examples:

- Matrix are the bounded linear operators between $\mathbb{R}^n$ or $\mathbb{C}^n$ spaces.

Bounded linear operators

Theorem

Given a Bounded linear operator A, its adjoint A^* is the only bounded linear operator such that $\langle A(x), y\rangle = \langle x, A^*(y)\rangle$ for any x, y.

Definition

A bounded linear operator $U : H_1 \to H_2$ is said to be unitary if $UU^* = I$ and $U^*U = I$, where I denotes the identity function. This is equivalent to U being surjective and satisfying

$$\langle Ux, Uy\rangle = \langle x, y\rangle$$

Unitary operators correspond to the isomorphisms in the category of Hilbert spaces, and play a main role in theory of representations.

Group Representations

Definition

A representation of a group G on a Hilbert space H is a group homomorphism $\rho : G \to B(H)$. ρ is said to be irreducible if there is no proper invariant subspace K of H such that $\rho(K) \subset K$.
Given a group G, the representations $\rho_1 : G \to B(H_1)$ and $\rho_2 : G \to B(H_2)$ are said to be unitarily equivalent if there is a unitary operator U such that for any $g \in G$, $U\rho_1(g) = \rho_2(g)U$.

$$\begin{array}{ccc} H_1 & \xrightarrow{U} & H_2 \\ {\scriptstyle \rho_1(g)}\big\downarrow & & \big\downarrow{\scriptstyle \rho_2(g)} \\ H_1 & \xrightarrow[U]{} & H_2 \end{array}$$

C*-algebras

Definition

A sixplet $(A, +, \times, \cdot, \|\cdot\|, *)$ is said to be a C*-algebra if the triplet $(A, +, \cdot)$ is a complex vector space, and the function $\times$ satisfies the identities

1. $(x \times y) \times z = x \times (y \times z)$
2. $(x + y) \times z = x \times z + y \times z$
3. $\alpha \cdot (x \times y) = (\alpha \cdot x) \times y = x \times (\alpha \cdot y)$ for any $\alpha \in \mathbb{C}$.

Furthermore the norm $\|\cdot\|$ satisfies $\|xy\| \leq \|x\|\|y\|$, and every Cauchy sequence in A converges to an element of A. The function $*$ is called an involution and satisfies

$$(x^*)^* = x \quad (xy)^* = y^*x^* \quad (\lambda \cdot x + y)^* = \overline{\lambda} \cdot x^* + y^* \quad \|xx^*\| = \|x^2\|$$

Theorem

Every C-algebra A is a sub C*-algebra of $B(H)$ for some Hilbert space H.*

Cuntz algebras

Definition

The Cuntz algebra $\mathcal{O}_n$ is the C*-algebra generated by the n isometries $\{s_1, s_2, \ldots, s_n\}$ satisfying

$$\sum_{j=1}^{n} s_j s_j^* = 1 \qquad s_i^* s_j = \delta_{ij}$$

for any $i, j \in \{1, 2, ..., n\}$.

Higman-Thompson groups

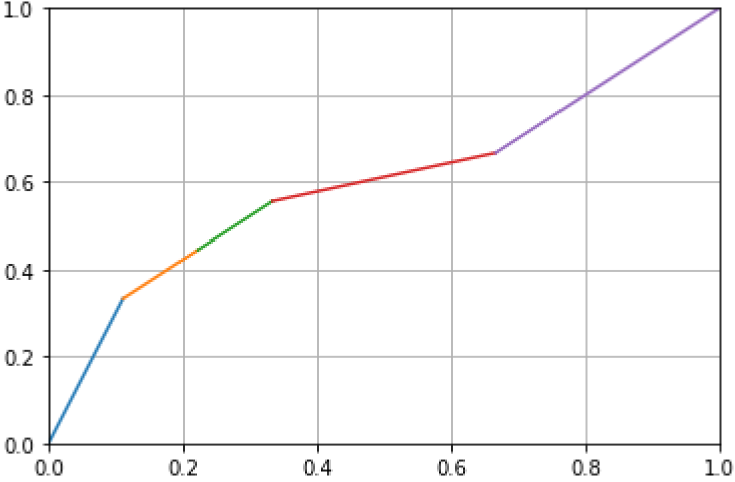
Definition

The Higman-Thompson group V_n is the group of piecewise linear functions $g : [0,1[\to [0,1[$ such that:

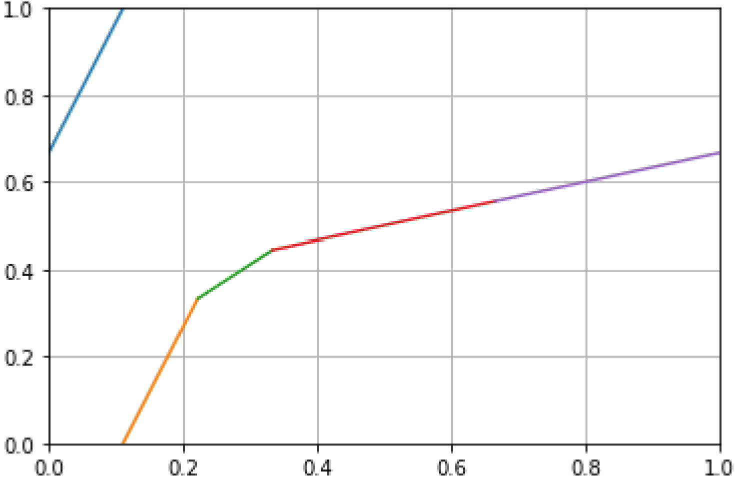
1. g is bijective in $[0,1[$.
2. $g(M) = M$.
3. $g'(x) = n^k$ for some k in the points where it is differentiable.
4. If g is not differentiable in x, then $x \in M$.

The Higman-Thompson group T_n is the subset of V_n such that g has at most one discontinuity. The Higman-Thompson group F_n is the subset of T_n such that g has no discontinuities.

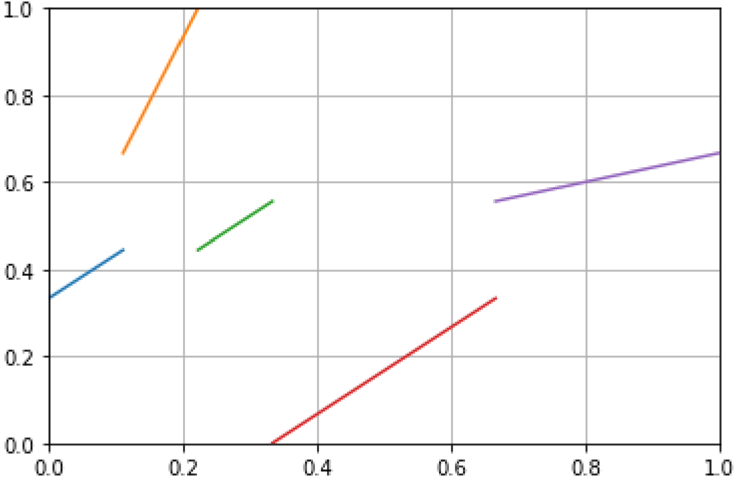
Higman-Thompson groups



Higman-Thompson groups



Higman-Thompson groups



Higman-Thompson groups

Definition

Let A be an alphabet with n letters. An admissible language $L := \{a_1, \ldots, a_m\}$ is a subset of A^* such that $A^\omega = \bigcup_{i=1}^m a_i A^\omega$ and no a_i is a prefix for a_j, for all $i, j \in \{1, \ldots, m\}$.
If $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_m\}$, we say that T is an admissible transformation

$$T = \begin{bmatrix} a_1 & a_2 & \ldots & a_m \\ b_1 & b_2 & \ldots & b_m \end{bmatrix},$$

The Higman-Thompson group V_n, is the group of all admissible transformations of a n-letters alphabet. We may assume, without loss of generality, that $b_1 < b_2 < \ldots < b_m$, where $\leq$ is the lexicographic order. In order for a table to be associated to a function that belongs to T_n, we additionally need to have: $a_i < a_{i+1} < \ldots < a_m < a_1 < \ldots < a_{i-1}$ for some $i \in \{1, 2, \ldots, m\}$. For the table to be associated to a function that belongs to F_n, we need to have $i = 1$.

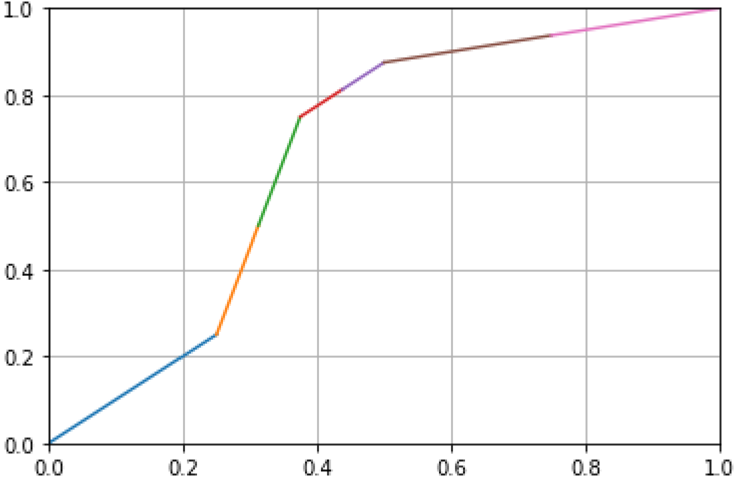
Examples:

$$\begin{bmatrix} 1 & \dots & i & \dots & n \\ 1 & \dots & n & \dots & i \end{bmatrix}$$

$$\begin{bmatrix} 1 & \dots & i1 & i2 & \dots & in & \dots & n1 & n2 & \dots & nn \\ 1 & \dots & n1 & n2 & \dots & nn & \dots & i1 & i2 & \dots & in \end{bmatrix}$$

$$\begin{bmatrix} 1 & 21 & 22 & 23 & 24 & 3 & 4 \\ 1 & 2 & 3 & 41 & 42 & 43 & 44 \end{bmatrix}$$

Higman-Thompson groups



Embeddings of the Algebra $\mathcal{O}_n$

Theorem

Given any integer $k \geq 1$, the Cuntz Algebra $\mathcal{O}_{k(n-1)+1}$ is embedded in $\mathcal{O}_n$. An embedding is the function $\iota : \mathcal{O}_{k(n-1)+1} \to \mathcal{O}_n$ satisfying:

$$\iota(\hat{s}_1) = s_1^k \qquad \iota(\hat{s}_{1+i(n-1)+(j-1)}) = \iota(\hat{s}_{i(n-1)+j}) = s_1^{k-i} s_j$$

for $0 \leq i < k$, $2 \leq j \leq n$.

proof:

$$s_1^k (s_1^*)^k + \sum_{i=1}^{k} \sum_{j=2}^{n} s_1^{k-i} s_j s_j^* (s_1^*)^{k-i} = 1$$

Embeddings of Higman-Thompson groups

Theorem

Let $\Psi_n : V_n \to \mathcal{O}_n$ be defined as

$$\Psi(g) = \Psi\left(\begin{bmatrix} a_1 & a_2 & \dots & a_m \\ b_1 & b_2 & \dots & b_m \end{bmatrix}\right) = s_{a_1}s_{b_1}^* + s_{a_2}s_{b_2}^* + \ldots + s_{a_m}s_{b_m}^*$$

Then Ψ is a faithful unitary representation of V_n in the group of unitaries of $\mathcal{O}_n$.

Our objective is that there is a group homomorphism E such that

$$\begin{array}{ccc} \mathcal{O}_{k(n-1)+1} & \xrightarrow{\iota} & \mathcal{O}_n \\ \uparrow \Psi_{k(n-1)+1} & & \downarrow \Psi_n^{-1} \\ V_{k(n-1)+1} & \xrightarrow[E]{} & V_n \end{array}$$

Embeddings of Higman-Thompson groups

Definition

Let $\gamma : Y \to X^*$ be the function such that, for all $y \in Y$:

$$\iota(\hat{s}_y) = s_{\gamma(y)}$$

Also, let $f : Y^* \to X^*$ be such that

$$f(u) = f(u_1 u_2 \dots u_m) = \gamma(u_1)\gamma(u_2)\dots\gamma(u_m)$$

. That is, the function such that:

$$\begin{aligned} \iota(\hat{s}_u) &= \iota(\hat{s}_{u_1 u_2 \dots u_m}) = \iota(\hat{s}_{u_1})\dots\iota(\hat{s}_{u_m}) \\ &= s_{\gamma(u_1)}\dots s_{\gamma(u_m)} = s_{\gamma(u_1)\gamma(u_2)\dots\gamma(u_m)} = s_{f(u)} \end{aligned}$$

So, for example, since $\iota(\hat{s_1}) = \underbrace{s_1 \dots s_1}_{k \text{ times}}$, we have $\gamma(1) = \underbrace{(1)\dots(1)}_{k \text{ times}}$.

Embeddings of Higman-Thompson groups

Theorem

The function E such that,

$$E : \begin{bmatrix} u_1 & u_2 & \dots & u_m \\ v_1 & v_2 & \dots & v_m \end{bmatrix} \mapsto \begin{bmatrix} f(u_1) & f(u_2) & \dots & f(u_m) \\ f(v_1) & f(v_2) & \dots & f(v_m) \end{bmatrix}$$

is an embedding of $V_{1+k(n-1)}$, $T_{1+k(n-1)}$ and $F_{1+k(n-1)}$ in V_n, T_n and F_n respectively.

Lemma

f is injective.

Corollary

If $f(a)$ is a prefix of $f(b)$, then a is a prefix of b.

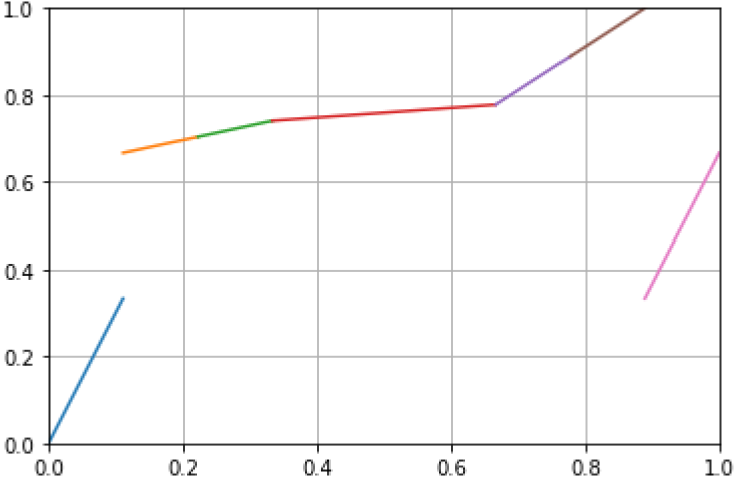
Proof Sketch: we prove that it sends elements of $V_{1+k(n-1)}$ to elements of V_n, this is:

$$Y^\omega = \bigcup_{i=1}^{m} u_i Y^\omega = \bigcup_{i=1}^{m} v_i Y^\omega \Rightarrow X^\omega = \bigcup_{i=1}^{m} f(u_i) X^\omega = \bigcup_{i=1}^{m} f(v_i) X^\omega$$

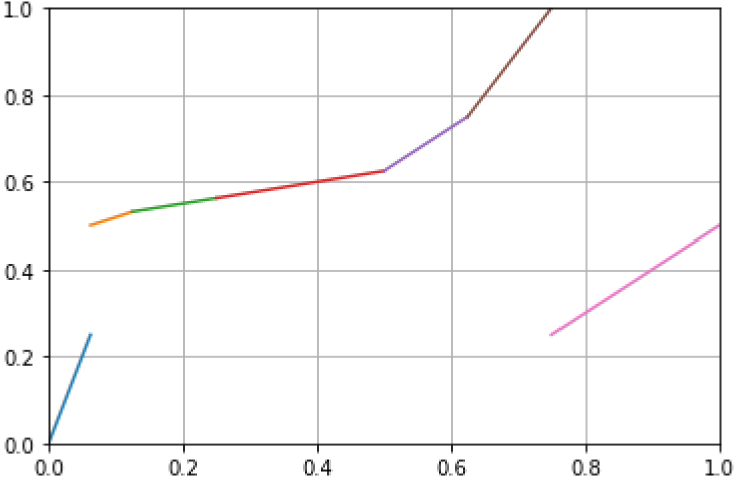
and that f preserves the lexicographic order,

$$a < b \Rightarrow f(a) < f(b)$$

Embeddings of Higman-Thompson groups



Embeddings of Higman-Thompson groups



Theorem

There is an embedding of V_2 in V_n. Given any $i, j \geq 2$ there are quasi-isometric embeddings from F_i to F_j. There is a quasi-isometric embedding from $T_{k(n-1)+1}$ to T_n, but that there are no embeddings from T_2 to T_n. If there is a non trivial embedding from $\mathcal{O}_m$ to $\mathcal{O}_n$, then m must be equal to $k(n-1)+1$, for some $k \geq 1$.

Theorem

Given a Hilbert space H and a representation $\pi : \mathcal{O}_n \to B(H)$, the function $\rho_\pi : V_n \to B(H)$:

$$\rho_\pi(g) = (\pi \circ \Psi)(g)$$

is a unitary representation of V_n in H.

The Hilbert space H_x

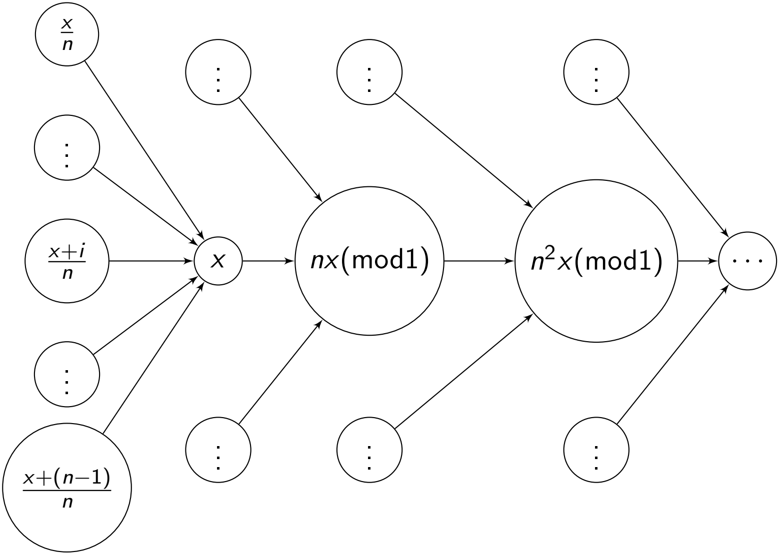
Definition

Fix $x \in [0,1[$. We define its orbit as $\mathrm{orb}(x) := \{f^z(x) : z \in \mathbb{Z}\}$, where $f(y) = ny \mod 1$. Note that $f^{-1}(x) = \{\frac{x}{n}, \frac{x+1}{n}, \ldots, \frac{x+(n-1)}{n}\}$,

$$f^{-2}(x) = \bigcup_{y \in f^{-1}(x)} f^{-1}(y)$$

and so on.

The Hilbert space H_x



The Hilbert space H_x

Theorem

Let $\sim$ be a binary relation in $[0,1[$ such that $y \sim x$ if and only if $y \in orb(x)$. This is equivalent to existing $a, b \in \mathbb{Z}$ such that $f^a(y) = f^b(x)$. $\sim$ is an equivalence relation whose equivalence classes are the different orbits.

Theorem

Denote by $H_x = \ell^2(orb(x))$. The set $\{\delta_y : y \in orb(x)\}$, where $\delta_y : orb(x) \to \mathbb{R}$ is the function

$$\delta_y(z) = \begin{cases} 1 & y = z \\ 0 & y \neq z \end{cases}$$

is an orthonormal basis of H_x.

The Hilbert space H_x

Definition

Define $S_i \in B(H_x)$ as

$$S_i \delta_y = \delta_{\frac{y+(i-1)}{n}}$$

Since these are linear bounded operators of a Hilbert space, they have an adjoint, which is given by

$$S_i^* \delta_y = \begin{cases} \delta_{yn-(i-1)} & y \in [\frac{i-1}{n}, \frac{i}{n}[\\ 0 & y \notin [\frac{i-1}{n}, \frac{i}{n}[\end{cases}$$

for the basis of H_x. One can easily verify that the operators S_i satisfy the Cuntz relations. We therefore have the following result.

Theorem

The function $\pi_x : \mathcal{O}_n \to B(H_x)$ such that $\pi_x(s_i) = S_i$ is a representation of $\mathcal{O}_n$ in H_x.

Representations ρ_x

Theorem

Given $g \in V_n$ and $y \in orb(x)$, we have

1. $g(y) \in orb(x)$
2. $\rho_x(g)(\delta_y) = \delta_{g(y)}$

Representations ρ_x

Lemma

Let $k \in orb(x)$. *Then, given* $m \in \mathbb{Z}$, $kn^m \mod 1 \in orb(x)$

Lemma

Let $k \in orb(x)$. *Then for any* $a \in \mathbb{N}$ *such that the number of digits in base* n *of* a *is* m, *we have that* $kn^{-m} + an^{-m} \in orb(x)$.

Lemma

Let $g \in V_n$, $f(x) = nx \mod 1$ *and* $orb(x) = \bigcup_{m \in \mathbb{Z}} \{f^m(x)\}$. *Then, for all* $y \in orb(x)$, $g(y) \in orb(x)$.

Representations ρ_x

Lemma

Let $y \in orb(x)$ *and* $u \in X^*$, *such that* $u = u_1 \dots u_k$. *Then* $S_u\delta_y = S_{u_1} \dots S_{u_k}\delta_y = \delta_a$, *where*

$$a = yn^{-k} + \sum_{i=1}^{k} \frac{u_i - 1}{n^i}.$$

Lemma

Let $y \in orb(x)$ *and* $v \in X^*$, *such that* $v = v_1 \dots v_m v_{m+1}$. *Then,* $y \in \phi(v)$ *if and only if* $b \in \phi(v_{m+1})$, *where*

$$b = n^m \left(y - \sum_{i=1}^{m} \frac{v_i - 1}{n^i} \right).$$

Representations ρ_x

Lemma

Let $y \in orb(x)$ and $v \in X^*$, such that $v = v_1 \dots v_m$ and $y \in \phi(v)$. Then $S_v^* \delta_y = \delta_b$, where

$$b = n^m \left(y - \sum_{i=1}^{m} \frac{v_i - 1}{n^i} \right).$$

Lemma

Let $v \in X^*$. If $y \notin \phi(v)$ then $S_v^* \delta_y = 0$.

Lemma

Let $y \in orb(x)$. Then, $\rho_x(g)\delta_y = \delta_{g(y)}$.

The operator U

Definition

Let $\iota' : \pi_x(\mathcal{O}_{k(n-1)+1}) \to B(H_x^{(k(n-1)+1)})$ be defined as

$$\iota'(\hat{S}_1) = S_1^k \qquad \iota'(\hat{S}_{i(n-1)+j}) = S_1^{k-i} S_j$$

for $0 \leq i < k$, $2 \leq j \leq n$.

$$\begin{array}{ccc}
\mathcal{O}_{k(n-1)+1} & \xleftarrow[\pi_x^{-1}]{} & \pi_x(\mathcal{O}_{k(n-1)+1}) \\
\Big\downarrow \iota & & \Big\downarrow \iota' \\
\mathcal{O}_n & \xrightarrow{\pi'_x} & \pi'_x(\mathcal{O}_n)
\end{array}$$

The operator U

Theorem

Let $y \in orb_m(x)$ for some $m \geq 2$ and $M = \{\hat{S}_1, \ldots, \hat{S}_m, \hat{S}_1^*, \ldots, \hat{S}_m^*\}$. Then there exist $T_1, T_2, \ldots, T_k \in M$ such that $\delta_y = T_1 T_2 \ldots T_k \delta_x$.

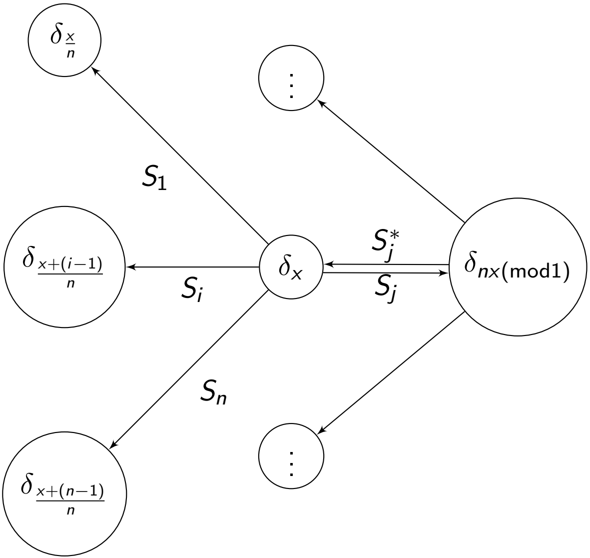
Theorem

Let $U : H_x^{(k(n-1)+1)} \to H_x^{(n)}$ be the function such that, given $\delta_y \in H_x^{(k(n-1)+1)}$

$$U(\delta_y) = U(T_1 \ldots T_k \delta_x) = \iota'(T_1 \ldots T_k)\delta_x.$$

Then U is a unitary operator.

The operator U



Main results

Theorem

Let $x \in [0,1[$. Then, $\pi_x : \mathcal{O}_{k(n-1)+1} \to B(H_x^{(k(n-1)+1)})$ and $\iota' \circ \pi_x : \mathcal{O}_{k(n-1)+1} \to B(U(H_x^{(k(n-1)+1)}))$ are unitarily equivalent.

$$\begin{array}{ccc} H_x^{(k(n-1)+1)} & \xrightarrow{U} & U(H_x^{(k(n-1)+1)}) \\ {\scriptstyle \pi_x(a)}\big\downarrow & & \big\downarrow{\scriptstyle \iota'(\pi_x(a))} \\ H_x^{(k(n-1)+1)} & \xrightarrow{U} & U(H_x^{(k(n-1)+1)}) \end{array}$$

Main results

Theorem

Given $x, y \in [0,1[$, the transformation $\pi_x : \mathcal{O}_{k(n-1)+1} \to B(H_x^{(k(n-1)+1)})$ is unitarily equivalent to $\pi_y : \mathcal{O}_{k(n-1)+1} \to B(H_y^{(k(n-1)+1)})$ if and only if $\iota' \circ \pi_x : \mathcal{O}_{k(n-1)+1} \to B(U(H_x^{(k(n-1)+1)}))$ is unitarily equivalent to $\iota' \circ \pi_y : \mathcal{O}_{k(n-1)+1} \to B(U(H_x^{(k(n-1)+1)}))$. Furthermore, π_x is irreducible in $H_x^{(k(n-1)+1)}$ if and only if $\iota' \circ \pi_x$ is irreducible in $U(H_x^{(k(n-1)+1)})$.

Proof:

$$\begin{array}{ccc} U_1(H_x^{(k(n-1)+1)}) & \xrightarrow{K} & U_2(H_y^{(k(n-1)+1)}) \\ \iota'(\pi_x(a)) \downarrow & & \downarrow \iota'(\pi_y(a)) \\ U_1(H_x^{(k(n-1)+1)}) & \xrightarrow{K} & U_2(H_y^{(k(n-1)+1)}) \end{array}$$

Main results

By the previous theorem, we obtain the following commutative diagram

$$\begin{array}{ccccccc}
U_1(H_x^{(k(n-1)+1)}) & \xrightarrow{U_1^*} & H_x^{(k(n-1)+1)} & \xrightarrow{K} & H_y^{(k(n-1)+1)} & \xrightarrow{U_2} & U_2(H_y^{(k(n-1)+1)}) \\
\downarrow{\scriptstyle \iota'(\pi_x(a))} & & \downarrow{\scriptstyle \pi_x(a)} & & \downarrow{\scriptstyle \pi_y(a)} & & \downarrow{\scriptstyle \iota'(\pi_y(a))} \\
U_1(H_x^{(k(n-1)+1)}) & \xrightarrow{U_1^*} & H_x^{(k(n-1)+1)} & \xrightarrow{K} & H_y^{(k(n-1)+1)} & \xrightarrow{U_2} & U_2(H_y^{(k(n-1)+1)})
\end{array}$$

Main results

Corollary

Let $x \in [0,1[$. Then, $\rho_x^{(k(n-1)+1)} : V_{k(n-1)+1} \to B(H_x^{(k(n-1)+1)})$ and $\rho_x^{(n)} \circ E : V_{k(n-1)+1} \to B(U(H_x^{(k(n-1)+1)}))$ are unitarily equivalent.

Corollary

Given $x, y \in [0,1[$, $\rho_x^{(k(n-1)+1)} \sim \rho_y^{(k(n-1)+1)}$ if and only if $\rho_x^{(n)} \circ E \sim \rho_y^{(n)} \circ E$. Also, $\rho_x^{(k(n-1)+1)}$ is an irreducible representation in $H_x^{(k(n-1)+1)}$ if and only if $\rho_x^{(n)} \circ E$ is irreducible in $U(H_x^{(k(n-1)+1)})$.

Main results

Using

Corollary

Let $x, y \in [0,1[$*. Then, given* $\iota' \circ \pi_x : \mathcal{O}_{k(n-1)+1} \to B(U(H_x^{(k(n-1)+1)}))$*,* $\iota' \circ \pi_y : \mathcal{O}_{k(n-1)+1} \to B(U(H_y^{(k(n-1)+1)}))$*, we have that* $\iota' \circ \pi_x \sim \iota' \circ \pi_y$ *if and only if* $x \sim y$*. Also,* $\iota' \circ \pi_x$ *is irreducible in* $U(H_x^{(k(n-1)+1)})$*.*

Theorem

Let $x, y \in [0,1[$*. Then, given* $\rho_x^{(n)} : V_n \to B(H_x^{(n)})$*,* $\rho_y^{(n)} : V_n \to B(H_y^{(n)})$*, we have that* $\rho_x^{(n)} \sim \rho_y^{(n)}$ *if and only if* $x \sim y$*.*

Main results

Theorem

Let $n \geq 2$. Then

$$C^*_{\rho_x^{(2)}}(E(V_n)) = \iota'(\pi_x(\mathcal{O}_n)).$$

Theorem

Let $n \geq 2$. Then, $\rho_x^{(n)}$ and $\rho_y^{(n)}$ are irreducible representations of V_n in $H_x^{(n)}$, and $\rho_x^{(n)} \sim \rho_y^{(n)}$ if and only if $x \sim y$. For any $k \geq 1$, $(\rho_x^{(n)} \circ E)$ is an irreducible representation of $V_{k(n-1)+1}$ in $U(H_x^{(k(n-1)+1)})$, a subspace of $H_x^{(n)}$. Furthermore, given another such representation $(\rho_y^{(n)} \circ E)$, $(\rho_x^{(n)} \circ E) \sim (\rho_y^{(n)} \circ E)$ if and only if $x \sim y$.