Brownian Motion and Levy's Theorem

Guilherme Varela

IST

3/09/2020

Levy's Theorem

Theorem (Levy's Theorem)

Let $X = \{X_t, \mathcal{F}_t, 0 \leq t < \infty\}$ *be a* ***continuous, adapted process*** *taking values in* $\mathbb{R}$ *such that the process*

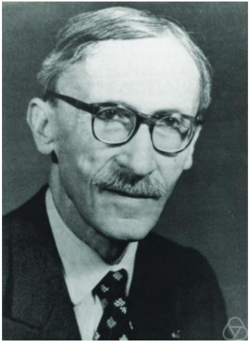
$$M_t = X_t - X_0, \quad 0 \leq t < \infty$$

is a ***continuous local martingale*** *relative to the filtration* $\{\mathcal{F}_t\}$*, and whose* ***quadratic variation*** *is given by*

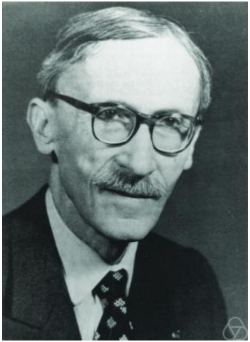
$$\langle M, M \rangle_t = t;$$

Then $\{X_t, \mathcal{F}_t, 0 \leq t < \infty\}$ *is a* ***Brownian Motion****.*

Paul Levy (1886-1971)

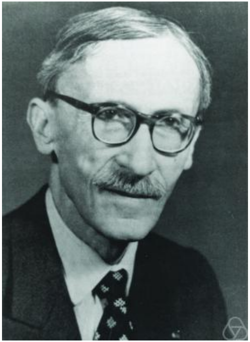


Paul Levy (1886-1971)



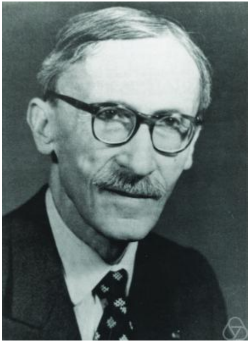
- **Levy's Continuity Theorem**
  - Convergence in Distribution of Random Variables $\leftrightarrow$ Pointwise Convergence of their Characteristic Functions

Paul Levy (1886-1971)



- **Levy's Continuity Theorem**
  - Convergence in Distribution of Random Variables $\leftrightarrow$ Pointwise Convergence of their Characteristic Functions
- **Characterization of the Brownian Motion**

Paul Levy (1886-1971)



- **Levy's Continuity Theorem**
  - Convergence in Distribution of Random Variables $\leftrightarrow$ Pointwise Convergence of their Characteristic Functions
- **Characterization of the Brownian Motion**
- **Many Other Topics (For Example Infinitely Divisible Laws)**

Probability Theory

- Probability Space $(\Omega, \mathcal{F}, \mathbb{P})$
- Filtration $\{\mathcal{F}_t, t \in \Gamma\}$, where $\Gamma = \mathbb{N}$ (discrete time) or $\Gamma = \mathbb{R}_0^+$ (continuous time)

Probability Theory

- Probability Space $(\Omega, \mathcal{F}, \mathbb{P})$
- Filtration $\{\mathcal{F}_t, t \in \Gamma\}$, where $\Gamma = \mathbb{N}$ (discrete time) or $\Gamma = \mathbb{R}_0^+$ (continuous time)

Definition (Usual Conditions)

A filtration $\mathcal{F}_t$ is said to have the usual conditions if it is right continuous and $\mathcal{F}_0$ contains all the $\mathbb{P}$-negligible events in $\mathcal{F}$.

Probability Theory

Definition (Measurable Process)

A stochastic process X is called measurable if, $\forall A \in \mathcal{B}(\mathbb{R}^d)$, the set $\{(t,\omega), X_t(\omega) \in A\}$ belongs to the product σ-field $\mathcal{B}(\mathbb{R}_0^+) \otimes \mathcal{F}$, or

$$(t,\omega) \to X_t(\omega) : (\mathbb{R}_0^+ \times \Omega, \mathcal{B}(\mathbb{R}_0^+) \otimes \mathcal{F}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$$

is measurable.

Probability Theory

Definition (Measurable Process)

A stochastic process X is called measurable if, $\forall A \in \mathcal{B}(\mathbb{R}^d)$, the set $\{(t, \omega), X_t(\omega) \in A\}$ belongs to the product σ-field $\mathcal{B}(\mathbb{R}_0^+) \otimes \mathcal{F}$, or

$$(t, \omega) \to X_t(\omega) : (\mathbb{R}_0^+ \times \Omega, \mathcal{B}(\mathbb{R}_0^+) \otimes \mathcal{F}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$$

is measurable.

Definition (Adapted Process)

A sequence of r.v. X_t is said to be adapted to $\mathcal{F}_t$ if $X_t \in \mathcal{F}_t, \forall t \in \Gamma$, which means that for every $A \in \mathcal{B}(\mathbb{R}^d)$, $\{\omega \in \Omega : X_t(\omega) \in A\} \in \mathcal{F}_t$.

Conditional Expectation

Given a σ-field $\mathcal{G}$ and an integrable random variable Y, $\mathbb{E}_{\mathcal{G}}(Y)$ must satisfy the conditions:

1. $\mathbb{E}_{\mathcal{G}}(Y) \in \mathcal{G}$
2. $\forall \Lambda \in \mathcal{G}$

$$\int_{\Lambda} Y(w)\mathbb{P}(dw) = \int_{\Lambda} \mathbb{E}_{\mathcal{G}}(Y)(w)\mathbb{P}(dw).$$

Conditional Expectation

Given a σ-field $\mathcal{G}$ and an integrable random variable Y, $\mathbb{E}_{\mathcal{G}}(Y)$ must satisfy the conditions:

1. $\mathbb{E}_{\mathcal{G}}(Y) \in \mathcal{G}$
2. $\forall \Lambda \in \mathcal{G}$

$$\int_{\Lambda} Y(w)\mathbb{P}(dw) = \int_{\Lambda} \mathbb{E}_{\mathcal{G}}(Y)(w)\mathbb{P}(dw).$$

- **(Geometric Perspective)**
  Suppose $\mathbb{E}(Y^2) < \infty$. Then $\mathbb{E}_{\mathcal{G}}(Y)$ is the random variable $X \in \mathcal{G}$ that minimizes $\mathbb{E}(Y - X)^2$.

Martingales

Let $\Gamma = \mathbb{N}$ and $S_n = X_1 + ... + X_n$.
To understand what information do we need from the past, our focus is to analyze $\mathbb{E}(S_{n+1}|S_n)$.

Martingales

Let $\Gamma = \mathbb{N}$ and $S_n = X_1 + ... + X_n$.
To understand what information do we need from the past, our focus is to analyze $\mathbb{E}(S_{n+1}|S_n)$.

- If X_n are independent r.v. with $\mathbb{E}(X_n) = 0$, we have $\mathbb{E}(S_{n+1}|S_n) = S_n$.

Martingales

Let $\Gamma = \mathbb{N}$ and $S_n = X_1 + ... + X_n$.
To understand what information do we need from the past, our focus is to analyze $\mathbb{E}(S_{n+1}|S_n)$.

- If X_n are independent r.v. with $\mathbb{E}(X_n) = 0$, we have $\mathbb{E}(S_{n+1}|S_n) = S_n$.
- This independence condition is much stronger than what we need!

Martingales

Let $\Gamma = \mathbb{N}$ and $S_n = X_1 + ... + X_n$.
To understand what information do we need from the past, our focus is to analyze $\mathbb{E}(S_{n+1}|S_n)$.

- If X_n are independent r.v. with $\mathbb{E}(X_n) = 0$, we have $\mathbb{E}(S_{n+1}|S_n) = S_n$.
- This independence condition is much stronger than what we need!
- Mathematically, martingales appeared to give exactly what we need! We only need that $\mathbb{E}(X_{n+1}|S_n) = 0$.

Martingales

Definition (Martingale)

A sequence of r.v. X_t is said to be a martingale with respect to a filtration $\{\mathcal{F}_t, t \in \mathbb{R}_0^+\}$ if it satisfies:

1. $\mathbb{E}(|X_t|) < \infty$ (integrability),
2. $X_t \in \mathcal{F}_t$ (measurability),
3. $\mathbb{E}_{\mathcal{F}_s}(X_t) = X_s, \quad \forall s \leq t$.

Also, if in the last equation we have $\leq$ or $\geq$, X_t is called a supermartingale or submartingale, respectively. The class of continuous martingales is denoted by $\mathcal{M}^c$.

Doob- Meyer Decomposition

Theorem (Doob-Meyer Decomposition)

Any submartingale X_t can be written in $X_t = M_t + A_t$, where $M_t \in \mathcal{M}^c$ and A_t is an increasing process.

Square Integrable Martingales

Definition (Square Integrable Martingale)

A continuous martingale $X = \{X_t, \mathcal{F}_t, 0 \le t < \infty\}$ is said to be square integrable if $\mathbb{E}(X_t)^2 < \infty, \forall t \ge 0$. The class of stochastic processes with this property is denoted by $\mathcal{M}_2^c$, where the c is for continuous.

Square Integrable Martingales

Definition (Square Integrable Martingale)

A continuous martingale $X = \{X_t, \mathcal{F}_t, 0 \le t < \infty\}$ is said to be square integrable if $\mathbb{E}(X_t)^2 < \infty, \forall t \geq 0.$ The class of stochastic processes with this property is denoted by $\mathcal{M}_2^c$, where the c is for continuous.

Definition (Quadratic Variation)

$\langle X \rangle_t$ is called the quadratic variation of X_t, where $\langle X \rangle_t$ is the adapted increasing process in the Doob -Meyer Decomposition of X_t^2. Or also, it is the unique process such that $X_t^2 - \langle X \rangle_t$ is a martingale.

If we consider a partition $0 = t_0 < t_1 < ... < t_n = t$ of $[0, t]$

$$\langle X \rangle_t = \lim_{n \to \infty} \sum_{k=1}^{n} (X_{t_k} - X_{t_{k-1}})^2.$$

Square Integrable Martingales

Definition (Cross Variation)

If $X, Y \in \mathcal{M}_2^c$, their cross variation is
$\langle X, Y \rangle_t = \frac{1}{4}[\langle X + Y \rangle_t - \langle X - Y \rangle_t]$.
And again, it can be thought of as

$$\langle X, Y \rangle_t = \lim_{n \to \infty} \sum_{k=1}^{n} (X_{t_k} - X_{t_{k-1}})(Y_{t_k} - Y_{t_{k-1}})$$

Local Martingales

Definition (Stopping and Optional Times)

Let T be a random time. Then T is a stopping time with respect to the filtration $\{\mathcal{F}_t, t \in \Gamma\}$ if the event $\{\omega \in \Omega : T(\omega) \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$.
A random time is an optional time if $\{\omega \in \Omega : T(\omega) < t\} \in \mathcal{F}_t$ for every $t \geq 0$.

Local Martingales

Definition (Stopping and Optional Times)

Let T be a random time. Then T is a stopping time with respect to the filtration $\{\mathcal{F}_t, t \in \Gamma\}$ if the event $\{\omega \in \Omega : T(\omega) \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$.
A random time is an optional time if $\{\omega \in \Omega : T(\omega) < t\} \in \mathcal{F}_t$ for every $t \geq 0$.

Definition (Local Martingale)

Let $X = \{X_t, \mathcal{F}_t, 0 \leq t < \infty\}$ be a continuous process with $X_0 = 0$ $\mathbb{P}$ -a.s. If there exists a nondecreasing sequence $\{T_n\}_{n=1}^{\infty}$ of stopping times of $\mathcal{F}_t$ such that $\{X_t^{(n)} = X_{t \wedge T_n}\}$ is a martingale for each $n \geq 1$ and $\mathbb{P}[lim_{n \longrightarrow \infty} T_n = \infty] = 1$, then we say that X is a continuous local martingale. It is usually written $X \in \mathcal{M}^{c,loc}$.

Brownian Motion

Brownian Motion

Consider the half-line $[0, \infty)$ and divide it in tiny intervals of length δ as it is shown below.

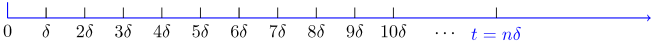


Figure 1: Dividing the half-line

Each one of these intervals corresponds to a time slot of length δ.

Brownian Motion

Consider the half-line $[0, \infty)$ and divide it in tiny intervals of length δ as it is shown below.

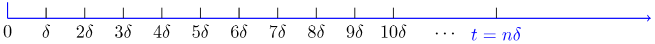


Figure 1: Dividing the half-line

Each one of these intervals corresponds to a time slot of length δ. Let us assume that we toss a fair coin and create the random variable for each interval $[i, i+1]$

$$X_i^\delta = \begin{cases} \sqrt{\delta}, & \text{with probability } 1/2 \\ -\sqrt{\delta}, & \text{with probability } 1/2 \end{cases}.$$

Brownian Motion

Note that X_i's are independent and

$$\mathbb{E}[X_i^\delta] = 0,$$
$$Var(X_i^\delta) = \delta.$$

Brownian Motion

Note that X_i's are independent and

$$\mathbb{E}[X_i^\delta] = 0,$$

$$Var(X_i^\delta) = \delta.$$

Therefore, we define the process $B_n^\delta(t)$ as

$$B_n^\delta(t) = B(n\delta) = \sum_{i=1}^{n} X_i^\delta.$$

Then $\forall t \in [0, \infty)$, as $n \to \infty$ and $\delta \to 0$

$$B_n^\delta \xrightarrow{dist} B(t) \sim N(0, t)$$

Brownian Motion

Note that X_i's are independent and

$$\mathbb{E}[X_i^\delta] = 0,$$

$$Var(X_i^\delta) = \delta.$$

Therefore, we define the process $B_n^\delta(t)$ as

$$B_n^\delta(t) = B(n\delta) = \sum_{i=1}^{n} X_i^\delta.$$

Then $\forall t \in [0, \infty)$, as $n \to \infty$ and $\delta \to 0$

$$B_n^\delta \xrightarrow{dist} B(t) \sim N(0, t)$$

Since the coin tosses are independent, we conclude that $B(t)$ has independent increments, which means that for $0 \leq t_1 < t_2 < ... < t_n$, the random variables

$$B(t_2) - B(t_1), B(t_3) - B(t_2), ..., B(t_m) - B(t_{m-1})$$

are independent.

Brownian Motion

Definition (Brownian Motion)

A standard, one dimensional Brownian Motion is a continuous process $B = \{B_t\}$ adapted to a filtration $\{\mathcal{F}_t, t \in \Gamma\}$ that satisfies the properties

- $B_0 = 0$ $\mathbb{P}$-a.s.,
- $B_t - B_s$ is independent of $\mathcal{F}_s$ for $0 \leq s \leq t$,
- $B_t - B_s$ is normally distributed with mean 0 and variance $t - s$,

which means it has independent and stationary increments.

Brownian Motion Sample Paths

If $B = \{B_t\}$ is a standard Brownian Motion, so are the following transformations:

Brownian Motion Sample Paths

If $B = \{B_t\}$ is a standard Brownian Motion, so are the following transformations:

1. (Scaling Property)

$$X_t = \frac{1}{\sqrt{a}} B_{at} \quad , a > 0$$

Brownian Motion Sample Paths

If $B = \{B_t\}$ is a standard Brownian Motion, so are the following transformations:

1. (Scaling Property)

$$X_t = \frac{1}{\sqrt{a}} B_{at} \quad , a > 0$$

2. (Time-Inversion)

$$Y_t = \begin{cases} tB_{\frac{1}{t}}, & 0 < t < \infty, \\ 0, & t = 0. \end{cases}$$

Brownian Motion Sample Paths

If $B = \{B_t\}$ is a standard Brownian Motion, so are the following transformations:

1. (Scaling Property)

$$X_t = \frac{1}{\sqrt{a}} B_{at} \quad , a > 0$$

2. (Time-Inversion)

$$Y_t = \begin{cases} tB_{\frac{1}{t}}, & 0 < t < \infty, \\ 0, & t = 0. \end{cases}$$

3. (Time-reversal)

$$Z_t = B_T - B_{T-t}, \;\; 0 \leq t \leq T.$$

Brownian Motion Sample Paths

If $B = \{B_t\}$ is a standard Brownian Motion, so are the following transformations:

1. (Scaling Property)

$$X_t = \frac{1}{\sqrt{a}} B_{at} \quad , a > 0$$

2. (Time-Inversion)

$$Y_t = \begin{cases} tB_{\frac{1}{t}}, & 0 < t < \infty, \\ 0, & t = 0. \end{cases}$$

3. (Time-reversal)

$$Z_t = B_T - B_{T-t}, \ \ 0 \leq t \leq T.$$

4. (Symmetry)

$$-B = \{-B_t\}.$$

Stochastic Calculus

How can we define

$$I_T(X) = \int_0^T X_t(\omega) dM_t(\omega)$$

for a general stochastic process X?

Stochastic Calculus

How can we define

$$I_T(X) = \int_0^T X_t(\omega) dM_t(\omega)$$

for a general stochastic process X?

Definition (Simple Process)

A process S is called simple if there exists a strictly increasing sequence of real numbers $\{t_n\}_{n=0}^{\infty}$ such that $t_0 = 0$ and $lim_{n \to \infty} t_n = \infty$, as well as a sequence of random variables $\{\epsilon_n\}_{n=0}^{\infty}$ with $sup_{n \geq 0} |\epsilon_n(\omega)| \leq C$, $\forall \omega \in \Omega$, $\epsilon_n \in \mathcal{F}_{t_n}$, such that

$$S_t(\omega) = \epsilon_0(\omega) 1_{\{0\}}(t) + \sum_{i=0}^{\infty} \epsilon_i(\omega) 1_{\{t_i, t_{i+1}\}}(t).$$

Stochastic Calculus

The natural path to proceed is to apply the martingale transform and define the integral of a simple process by

$$I_t(S) = \sum_{i=0}^{\infty} \epsilon_i (M_{t \wedge t_{i+1}} - M_{t \wedge t_i}), \;\; 0 \leq t < \infty.$$

Properties of the Integral

Properties of the Integral

$$I_0(X) = 0 \ \ \mathbb{P}\text{-a.s.}$$

Properties of the Integral

$$I_0(X) = 0 \;\; \mathbb{P}\text{-a.s.}$$

$$\text{(Linearity)} \quad I(\alpha X + \beta Y) = \alpha I(X) + \beta I(Y) \quad \forall \alpha, \beta \in \mathbb{R}$$

Properties of the Integral

$$I_0(X) = 0 \ \ \mathbb{P}\text{-a.s.}$$

$$\text{(Linearity)} \quad I(\alpha X + \beta Y) = \alpha I(X) + \beta I(Y) \quad \forall \alpha, \beta \in \mathbb{R}$$

$$\text{(Martingale Property)} \quad \mathbb{E}[I_t(X) | \mathcal{F}_s] = I_s(X) \ \ \mathbb{P}\text{-a.s.}$$

Itô's Rule

Itô's Rule

Definition (Semimartingale)

A continuous semimartingale $X = \{X_t, \mathcal{F}_t, 0 \le t < \infty\}$ is an adapted process which has the decomposition $\mathbb{P}$-a.s.

$$X_t = X_0 + M_t + B_t; \quad 0 \le t < \infty$$

where $M = \{M_t, \mathcal{F}_t, 0 \le t < \infty\} \in \mathcal{M}^{c,loc}$ and $B = \{B_t\}$ is the difference of continuous non decreasing, adapted processes

$$B_t = A_t^+ - A_t^- .$$

This decomposition should be the minimal decomposition of B, i.e, A_t^+ and A_t^- are the positive and negative variations of B on $[0, t]$.

Itô's Rule

Theorem (Itô's Rule)

Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class C^2 and let $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a continuous semimartingale. Then $\mathbb{P}$-a.s.

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dM_s + \int_0^t f'(X_s)dB_s + \frac{1}{2}\int_0^t f''(X_s)d\langle M\rangle_s,$$

with $0 \leq t < \infty$. The above equality can be also be written in differential form:

$$d(f(X_t)) = f'(X_t)dM_t + f'(X_t)dB_t + \frac{1}{2}f''(X_t)d\langle M\rangle_t.$$

Examples

Let $\{B_t : t \geq 0\}$ a Brownian Motion. Then when considering $f(x) = x^2/2$ and $g(x) = x^3/3$ we have from Itô's Rule:

$$\int_0^t B_s dB_s = \frac{1}{2}(B_t^2 - t)$$

and

$$\int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds.$$

For f, applying the formula above we have

$$\frac{B_t^2}{2} = \frac{B_0^2}{2} + \int_0^t B_s dB_s + \int_0^t d\langle B \rangle_s.$$

Using the fact $B_0 = 0$ $\mathbb{P}$a.s. and the properties of Brownian Motion we have the result.

Multidimensional Formula

Theorem (Multidimensional Formula)

Let $\{M_t = (M_t^{(1)}, ..., M_t^{(d)}\}$ *with* $M_t^{(i)} \in \mathcal{M}^{c,loc}$ *for* $1 \leq i \leq d$ *and* $\{B_t = (B_t^{(1)}, ..., B_t^{(d)})\}$ *a vector of adapted processes of bounded variation with* $B_0 = 0 \mathbb{P} a.s.$
Let $X_t = X_0 + M_t + B_t$, *where* $X_0 \in \mathcal{F}_0$. *Let* $f(t,x) : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ *be of class* $C^{1,2}$. *Then,* $\mathbb{P}$*-a.s.*

$$f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial}{\partial t} f(s, X_s) ds + \sum_{i=1}^{d} \int_0^t \frac{\partial}{\partial x_i} f(s, X_s) dB_s^{(i)}$$
$$+ \sum_{j=1}^{d} \int_0^t \frac{\partial}{\partial x_l} f(s, X_s) dM_s^{(j)}$$
$$+ \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} f(s, X_s) d\langle M^{(i)}, M^{(j)} \rangle_s, \quad 0 \leq t < \infty.$$

Levy's Theorem

Theorem (Levy's Theorem)

Let $X = \{X_t = (X_t^{(1)}, ..., X_t^{(d)}), \mathcal{F}_t, 0 \le t < \infty\}$ *be a continuous, adapted process in* $\mathbb{R}^d$ *such that, for every component* $1 \le k \le d$*, the process*

$$M_t^{(k)} = X_t^{(k)} - X_0^{(k)}, \quad 0 \le t < \infty$$

is a continuous local martingale relative to the filtration $\{\mathcal{F}_t, t \in \Gamma\}$*, and whose cross variations are given by*

$$\langle M^{(k)}, M^{(j)} \rangle_t = \delta_{kj} t; \quad 1 \le k, j \le d$$

Then $\{X_t, \mathcal{F}_t, 0 \le t < \infty\}$ *is a d-dimensional Brownian Motion.*

Levy's Theorem

- What we need to prove is that for $0 \leq s < t$, the random vector $X_t - X_s$ is independent of $\mathcal{F}_s$ and has a d-variate normal distribution with mean zero and covariance matrix equal to $(t - s) \times I_d$.

Levy's Theorem

- What we need to prove is that for $0 \leq s < t$, the random vector $X_t - X_s$ is independent of $\mathcal{F}_s$ and has a d-variate normal distribution with mean zero and covariance matrix equal to $(t-s) \times I_d$.

- But now, from Levy's continuity theorem, it suffices to prove that for $u \in \mathbb{R}^d$

$$\mathbb{E}[e^{i(u.(X_t - X_s))}|\mathcal{F}_s] = e^{-\frac{1}{2}\|u\|^2(t-s)}, \quad \mathbb{P} - a.s.$$

Levy's Theorem

- What we need to prove is that for $0 \leq s < t$, the random vector $X_t - X_s$ is independent of $\mathcal{F}_s$ and has a d-variate normal distribution with mean zero and covariance matrix equal to $(t - s) \times I_d$.

- But now, from Levy's continuity theorem, it suffices to prove that for $u \in \mathbb{R}^d$
$$\mathbb{E}[e^{i(u.(X_t - X_s))}|\mathcal{F}_s] = e^{-\frac{1}{2}\|u\|^2(t-s)}, \quad \mathbb{P} - a.s.$$

- When we fix $u = (u_1, ..., u_d) \in \mathbb{R}^d$, the function $f(x) = e^{i(u.x)}$ satisfies
$$\frac{\partial}{\partial x_j} f(x) = iu_j f(x), \;\; \frac{\partial^2}{\partial x_j \partial x_k} f(x) = -u_j u_k f(x).$$

Levy's Theorem

- Therefore, using the multidimensional formula for the real and imaginary parts of f, we obtain
  1. $\mathcal{R}(e^{i(u.X_t)}) = \mathcal{R}(e^{i(u.X_s)}) - \frac{1}{2}\sum_{j=1}^{d} u_j^2 \int_s^t e^{i(u.X_v)} dv$
  2. $\mathcal{I}(e^{i(u.X_t)}) = \mathcal{I}e^{i(u.X_s)} + \sum_{j=1}^{d} u_j \int_s^t e^{i(u.X_v)} dM_v^{(j)}$

  and we have:

$$e^{i(u.X_t)} = e^{i(u.X_s)} + i\sum_{j=1}^{d} u_j \int_s^t e^{i(u.X_v)} dM_v^{(j)} - \frac{1}{2}\sum_{j=1}^{d} u_j^2 \int_s^t e^{i(u.X_v)} dv.$$

Levy's Theorem

- Note that $|f(x)| \leq 1 \ \forall x \in \mathbb{R}^d$ and since $\langle M^{(j)} \rangle_t = t$, this implies that $M^{(j)} \in \mathcal{M}_2^c$. Hence, the real and imaginary parts of $\{\int_0^t e^{i(u.X_v)} dM_v^{(j)}, \mathcal{F}_t; 0 \leq t < \infty\}$ are not only in $\mathcal{M}^{c,loc}$ but also in $\mathcal{M}_2^c$.

Levy's Theorem

- Note that $|f(x)| \leq 1 \ \forall x \in \mathbb{R}^d$ and since $\langle M^{(j)} \rangle_t = t$, this implies that $M^{(j)} \in \mathcal{M}_2^c$. Hence, the real and imaginary parts of $\{\int_0^t e^{i(u.X_v)} dM_v^{(j)}, \mathcal{F}_t; 0 \leq t < \infty\}$ are not only in $\mathcal{M}^{c,loc}$ but also in $\mathcal{M}_2^c$.
- Consequently

$$\mathbb{E}\Big[\int_s^t e^{i(u.X_v)} dM_v^{(j)} \Big| \mathcal{F}_s\Big] = 0, \qquad \mathbb{P} - a.s.$$

Levy's Theorem

- Note that $|f(x)| \leq 1 \ \forall x \in \mathbb{R}^d$ and since $\langle M^{(j)} \rangle_t = t$, this implies that $M^{(j)} \in \mathcal{M}_2^c$. Hence, the real and imaginary parts of $\{\int_0^t e^{i(u.X_v)} dM_v^{(j)}, \mathcal{F}_t; 0 \leq t < \infty\}$ are not only in $\mathcal{M}^{c,loc}$ but also in $\mathcal{M}_2^c$.
- Consequently

$$\mathbb{E}\Big[\int_s^t e^{i(u.X_v)} dM_v^{(j)} \Big| \mathcal{F}_s\Big] = 0, \qquad \mathbb{P} - a.s.$$

- Now, for each $A \in \mathcal{F}_s$ and multiplying the last equation by $e^{-i(u.X_s)} \mathbb{1}_A$ and applying expected values we obtain

$$\mathbb{E}[e^{(u.(X_t - X_s))} \mathbb{1}_A] = \mathbb{P}(A) - \frac{1}{2}\|u\|^2 \int_s^t \mathbb{E}[e^{(u.(X_v - X_s))} \mathbb{1}_A] dv$$

  which is an integral equation already solved with solution :

$$\mathbb{E}[e^{(u.(X_t - X_s))} \mathbb{1}_A] = \mathbb{P}(A) e^{-\frac{1}{2}\|u\|^2 (t-s)}, \qquad \forall A \in \mathcal{F}_s.$$

References

- I.Karatzas and S.E.Shreve, Brownian Motion and Stochastic Calculus, Springer-Verlag(1987)
- J.M.Steele, Stochastic Calculus and Financial Applications,Springer (2000)
- R.Durret, Probability: Theory and Examples, Version 5, January 11,2019
- K.L.Chung, A Course In Probability Theory,Academic Press, Third Edition (2001)
- P.Billingsley, Convergence of Probability Measures,John Wiley Sons,Inc, Second Edition 1999