Fotografias de rectas por câmaras catadióptricas

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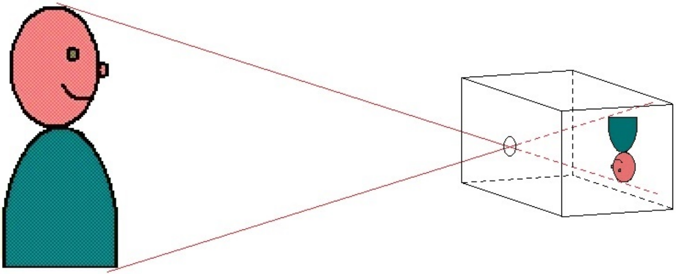
2020

Câmara Estenopeica



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  - φ - plano da fotografia
  - O - foco

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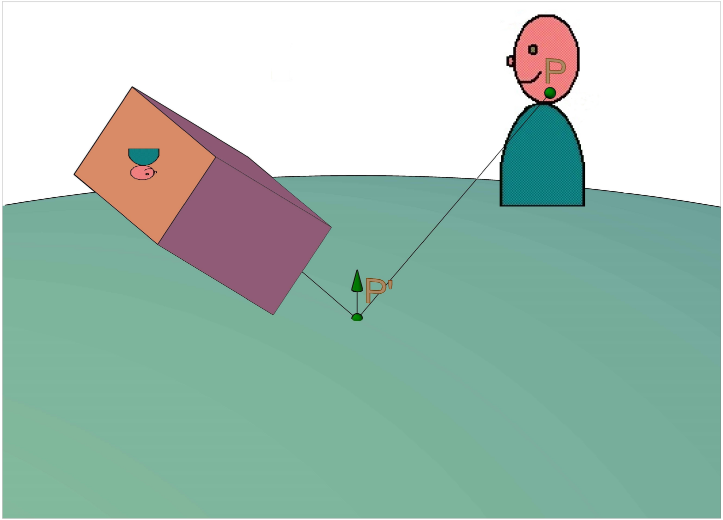
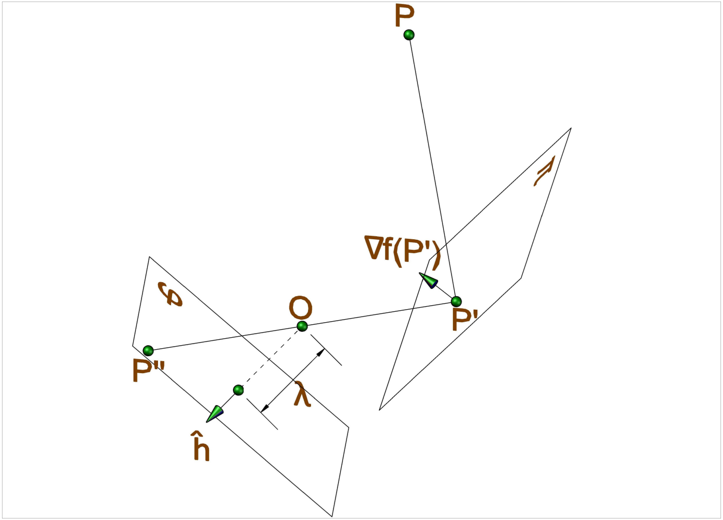
- Os pontos, P', pertencentes ao plano paralelo a φ que passa por O, não vão ser fotografados.


P
P'

Câmara Catadióptrica



Caracterização do espelho

- Consideremos que
  - S é o conjunto de pontos que constituem o espelho
  - existe $f : \mathbb{R}^3 \to \mathbb{R}$, diferenciável, tal que

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- Denotemos
  - π - plano tangente a S no ponto P'
  - n - a recta normal a S no ponto P'

Axioma 1 (1ª lei da reflexão)

Os raios incidente, reflectido, e a recta n, são complanares.

Leis da Reflexão

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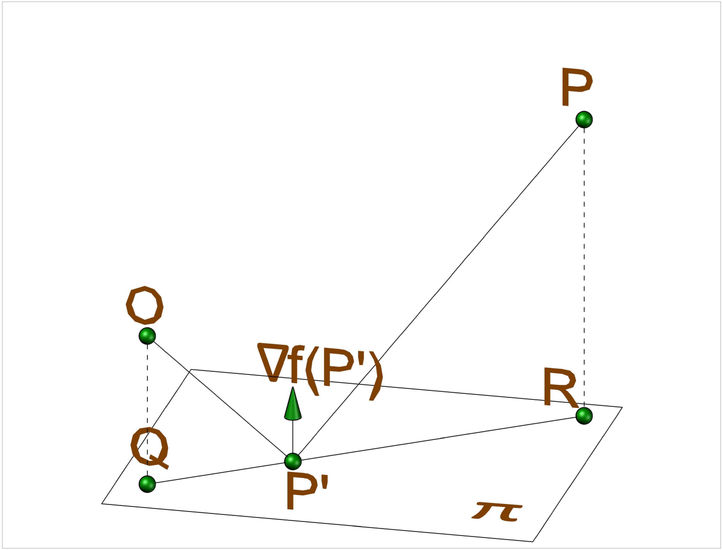
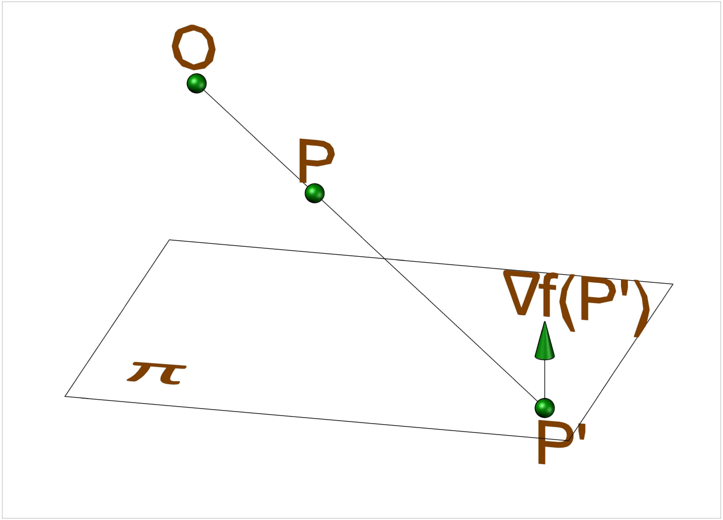
Os raios incidente, reflectido, e a recta n, são complanares.

Axioma 2 (2ª lei da reflexão)

O ângulo entre o raio incidente e a recta n, é igual ao ângulo entre o raio reflectido e a mesma recta n.

Leis da Reflexão

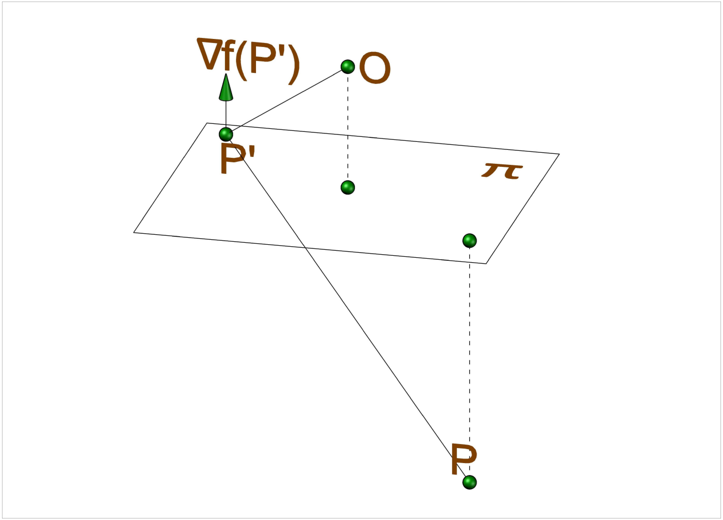



O
P
∇f(P')
π
P'

Axioma 3

Se o raio incidente não for colinear com a recta n, então os raios incidente e reflectido não são colineares

Leis da Reflexão



Leis da Reflexão

Axioma 4

O ponto fotografado, P, e o foco, O, estão no mesmo semiespaço definido pelo plano π

Imagem no Espelho

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A imagem em S de um ponto P, por O, é o conjunto de pontos P' pertencentes a S, tais que o triplo (O, P, P') respeita os axiomas 1, 2 e 3.

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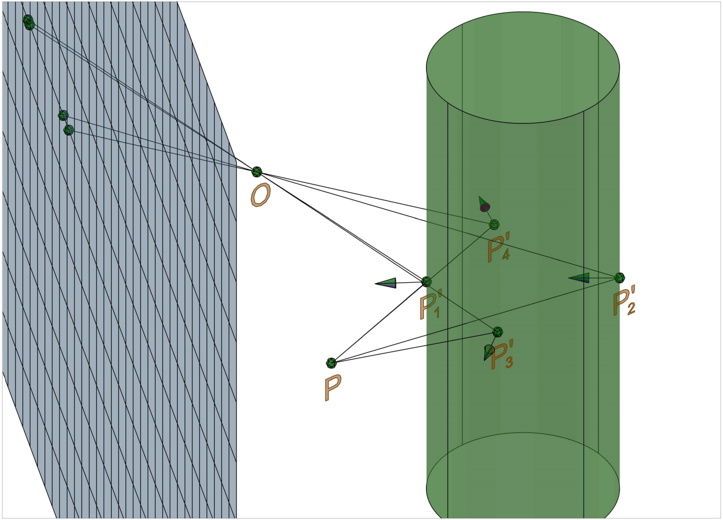
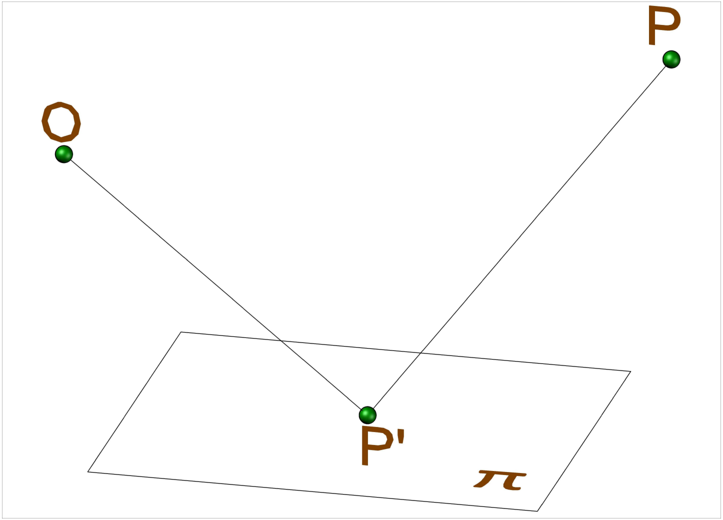
Imagem reflectora no Espelho

A imagem reflectora em S de um ponto P, por O, é o conjunto de pontos P' pertencentes a S, tais que o triplo (O, P, P') respeita os axiomas 1, 2, 3 e 4.

Imagem no Espelho



- Caracterização das imagens no espelho



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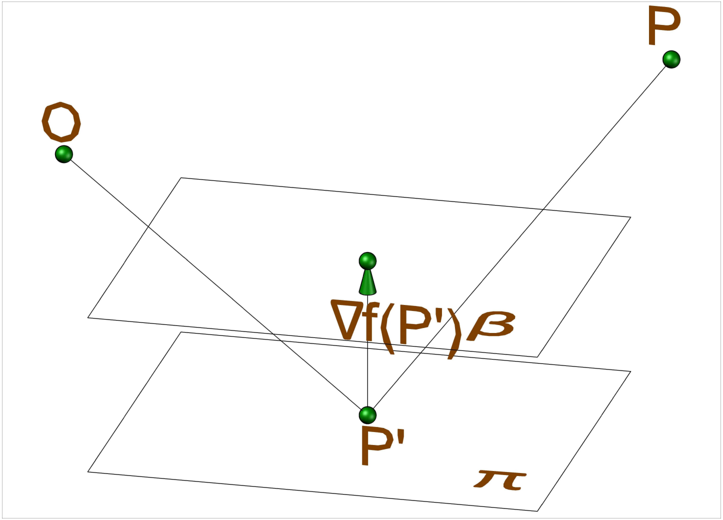

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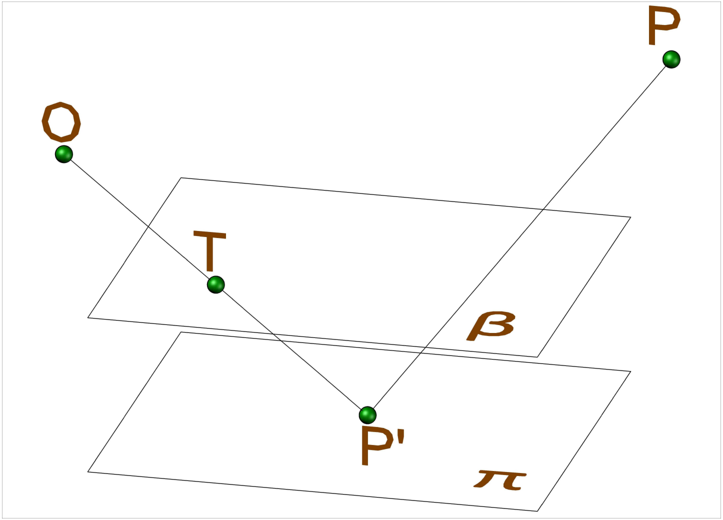

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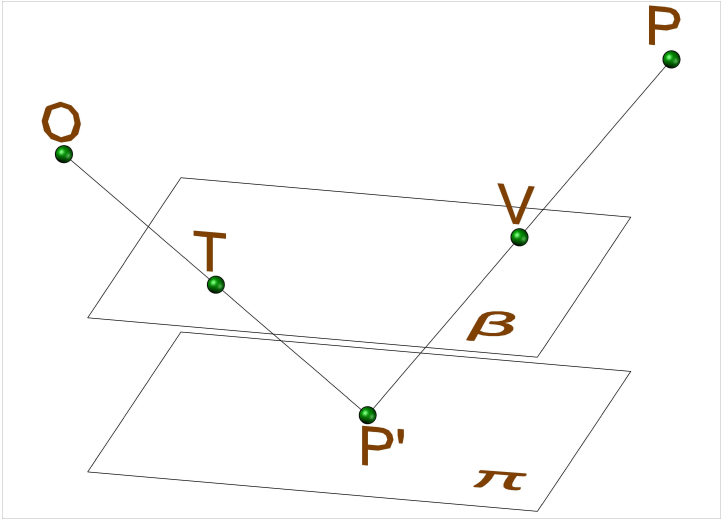

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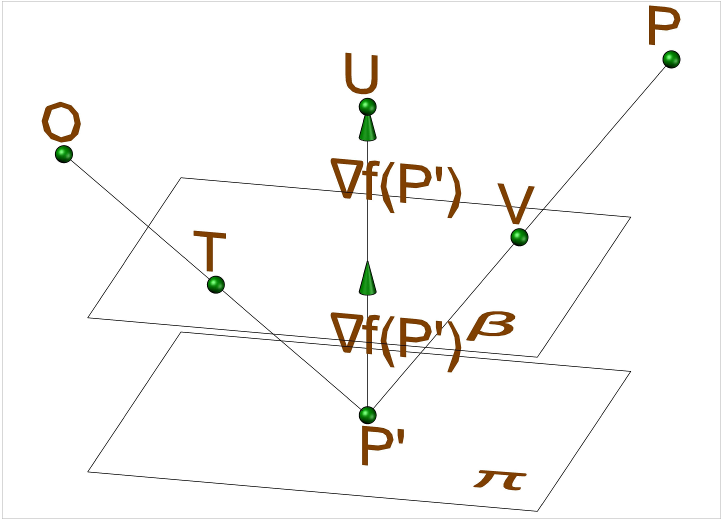

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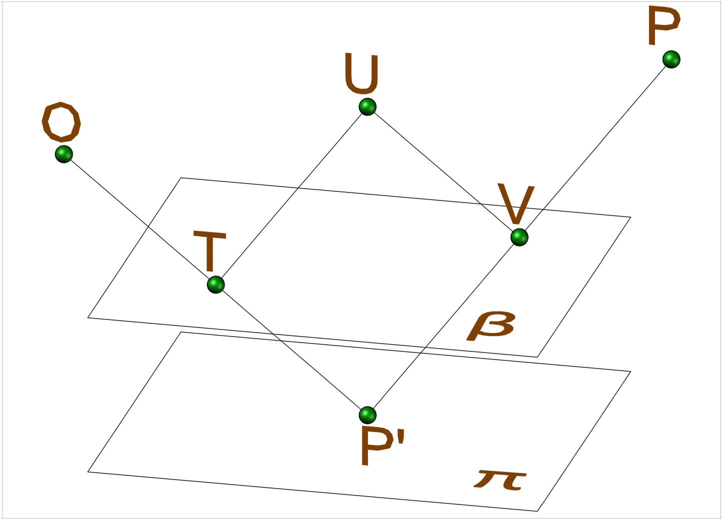

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$$\frac{2\nabla f(P')}{\langle \nabla f(P'), \nabla f(P') \rangle} = \frac{O - P'}{\langle \nabla f(P'), O - P' \rangle} + \frac{P - P'}{\langle \nabla f(P'), P - P' \rangle}$$

Imagem num Espelho Algébrico

- Designemos

$$\begin{bmatrix} g_1(X,P) \\ g_2(X,P) \\ g_3(X,P) \end{bmatrix} = \left\langle \nabla f(X), \overrightarrow{PX} \right\rangle \langle \nabla f(X), \nabla f(X) \rangle \overrightarrow{OX} + \left\langle \nabla f(X), \overrightarrow{OX} \right\rangle \langle \nabla f(X), \nabla f(X) \rangle \overrightarrow{PX} - 2 \left\langle \nabla f(X), \overrightarrow{OX} \right\rangle \left\langle \nabla f(X), \overrightarrow{PX} \right\rangle \nabla f(X)$$

$$h(X) = \left\langle \nabla f(X), \overrightarrow{OX} \right\rangle$$

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- P' é uma imagem de P, no espelho, se e só se,

$$f(P') = g_1(P', P) = g_2(P', P) = g_3(P', P) = 0$$

$$h(P') \neq 0, \; j(P') \neq 0$$

Imagem num Espelho Algébrico

- Denotemos

$$I_1[P,P'] = \langle f, g_1, g_2, g_3\rangle = \{pf + p_1 g_1 + p_2 g_2 + p_3 g_3,\ p, p_1, p_2, p_3 \in \mathbb{R}[X,P]\}$$

$$I_2\left[P,P'\right] = \langle hj\rangle = \{phj,\ p \in \mathbb{R}[X,P]\} \subseteq \mathbb{R}\left[P,P'\right]$$

$$V_1 = V(I_1) = \left\{(X,P)\in\mathbb{R}^6 : f(X) = g_1(X,P) = g_2(X,P) = g_3(X,P) = 0\right\}$$

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- O conjunto de pontos que são imagem de P na superfície do espelho é $V_1\setminus V_2$
- O fecho algébrico deste conjunto, é a menor variedade algébrica que o contém

- Considerado a operação *saturação*

$$J = I_1 : \langle hj \rangle^{\infty} = \left\{ q \in \mathbb{R}[X] : \ \exists k \in \mathbb{N} \ tal \ que \ q(hj)^k \in I_1 \right\}$$

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- Uma proposição encontrada na literatura, indica que o fecho algébrico de $V_1 \setminus V_2$ é

$$\overline{V_1 \setminus V_2} = V(J) = \{X \in \mathbb{R}^n : j(X) = 0, \forall j \in J\}$$

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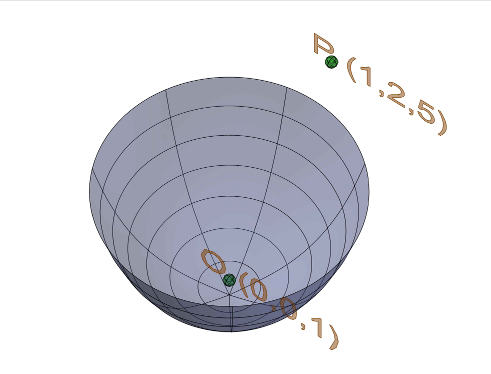
- A operação *saturação* é possível realizar computacionalmente para exemplos concretos, moderadamente complexos

- Exemplo 1



$$f(X, Y, Z) = X^2 + Y^2 - 2Z + 1, \quad O = (0, 0, 1), \quad P = (1, 2, 5)$$

Imagem num Espelho Algébrico

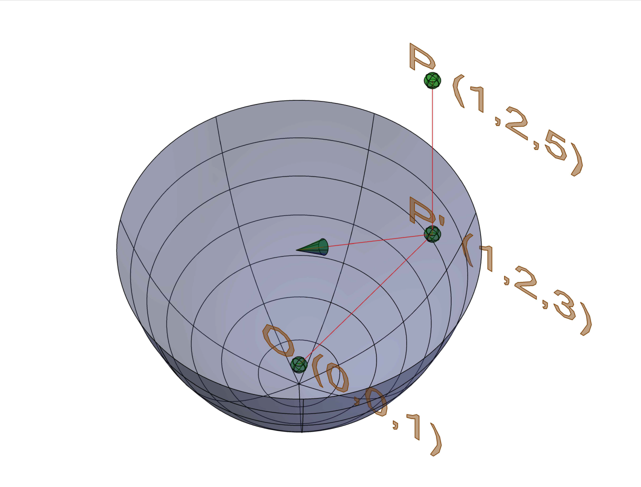
$$I_1\left[\mathbf{P},P'\right]=\left\langle\begin{array}{c}x^2+y^2-2z+1,\\ 16x^3y-8x^2y^2+16xy^3-8y^4+16x^3z+16xy^2z-32x^3-32xy^2-8x^2z-\\ 32xyz+8y^2z-16xz^2-8x^2+16xy-16y^2+80xz-32x+8z-8,\\ -16x^4+8x^3y-16x^2y^2+8xy^3+16x^2yz+16y^3z-32x^2y-32y^3+\\ 16x^2z-16xyz-16y^2z-16yz^2-32x^2+8xy-16y^2+80yz-32y+16z-16,\\ 16x^4z+32x^2y^2z+16y^4z-32x^4-64x^2y^2-32y^4-8x^3z-16x^2yz-\\ 8xy^2z-16y^3z-16x^2z^2-16y^2z^2-8x^3-16x^2y-8xy^2-16y^3+\\ 80x^2z+80y^2z-32x^2-32y^2+8xz+16yz-8z-16y\end{array}\right\rangle$$

$$I_2\left[\mathbf{P},P'\right]=\left\langle\left(4x^2+4y^2+4\right)\left(-2x^2-2y^2+2z-2\right)\right\rangle$$

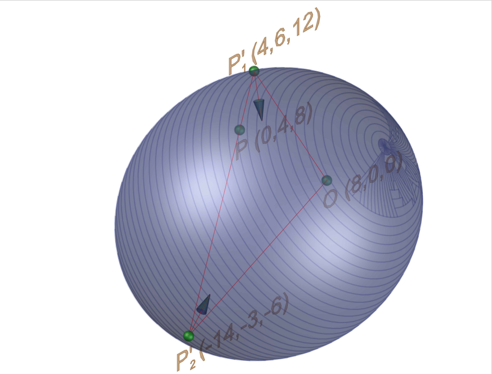
$$J\left[P'\right]=I_1:I_2^\infty=\langle z-3,y-2,x-1\rangle$$

$$\overline{V_1\setminus V_2}=V(J)=\{(1,2,3)\}$$

Imagem num Espelho Algébrico

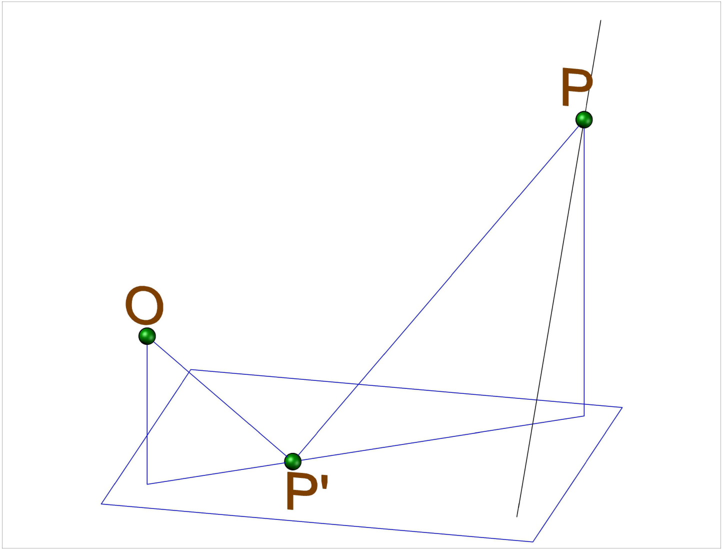


- Exemplo 2



$$f(X,Y,Z) = 3X^2 + 4Y^2 + 4Z^2 - 768, \quad O = (8,0,0), \quad P = (0,4,8)$$

- Para um espelho plano



Fotografias de rectas

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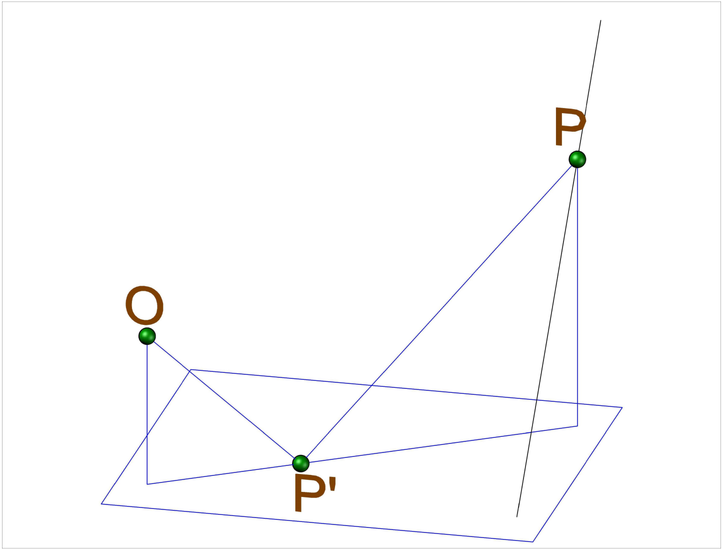

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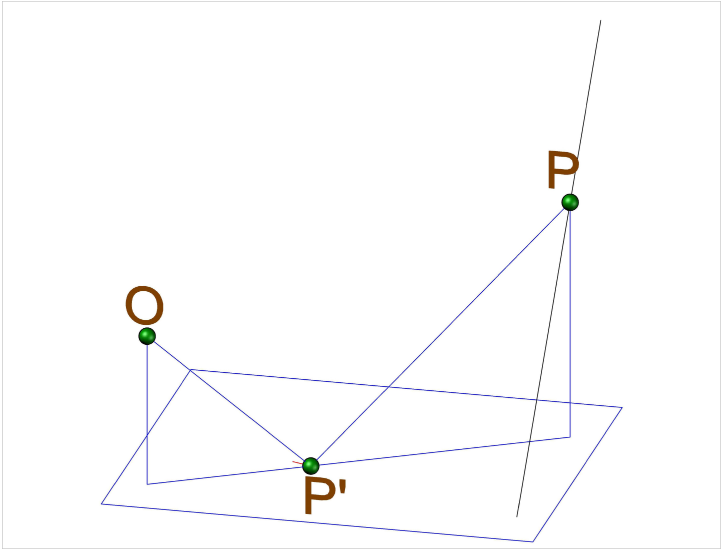

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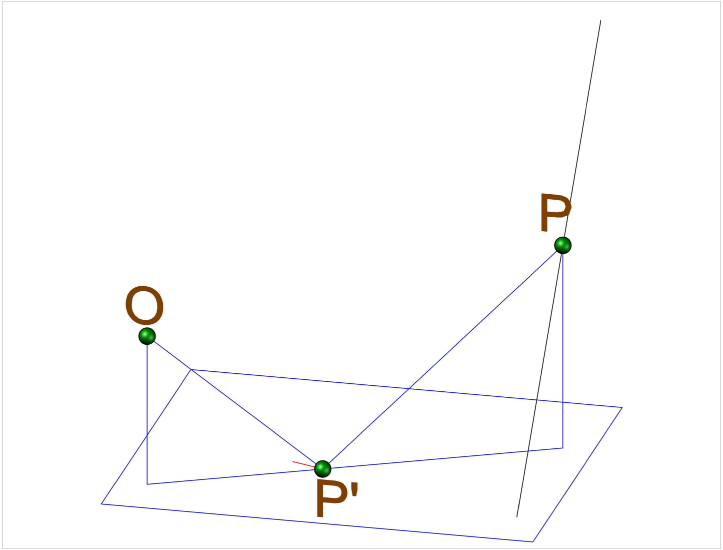

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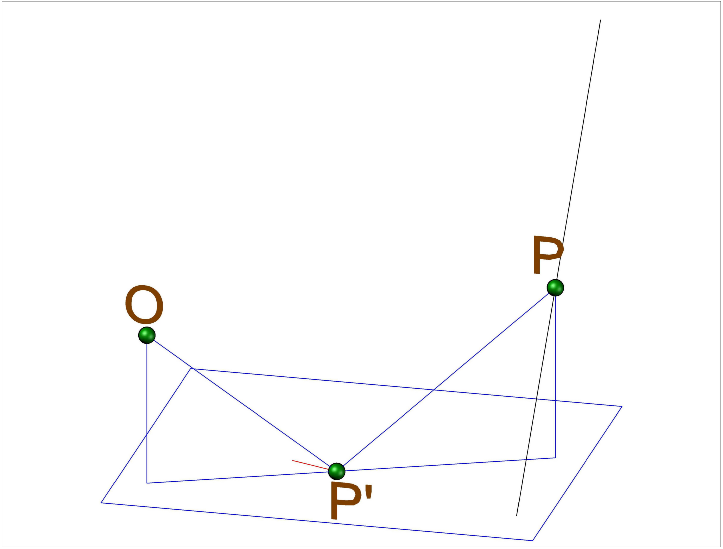

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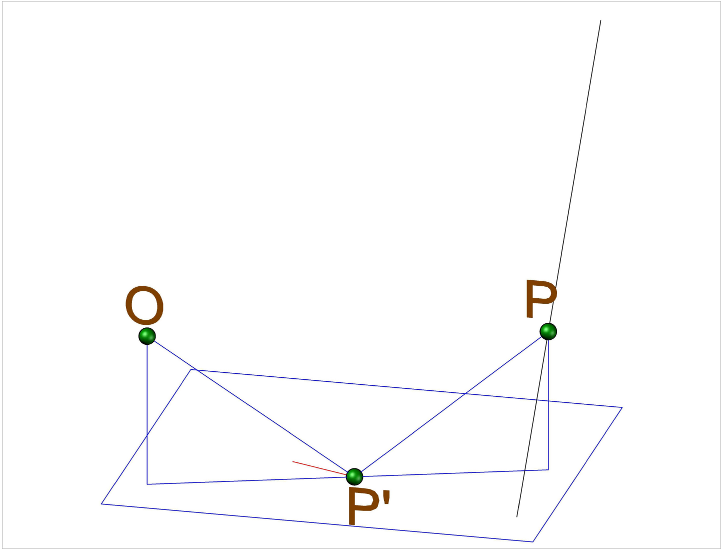

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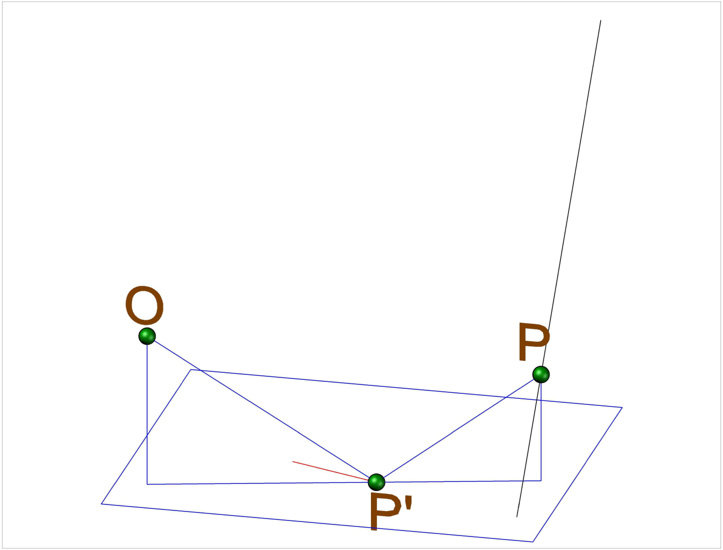

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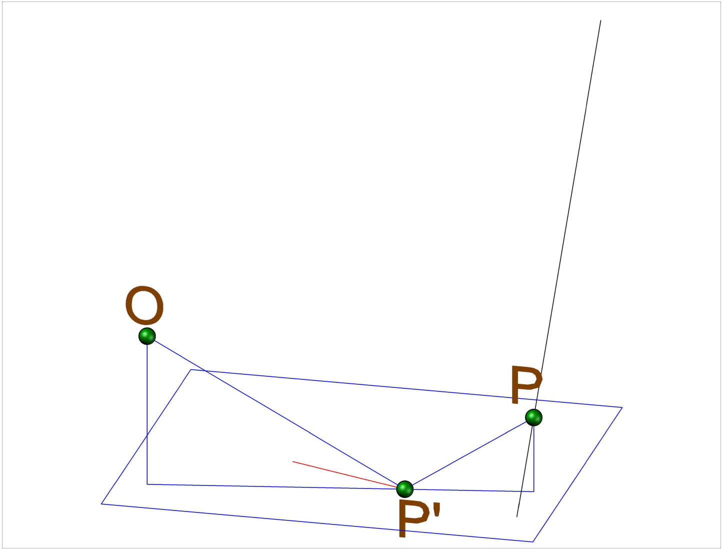

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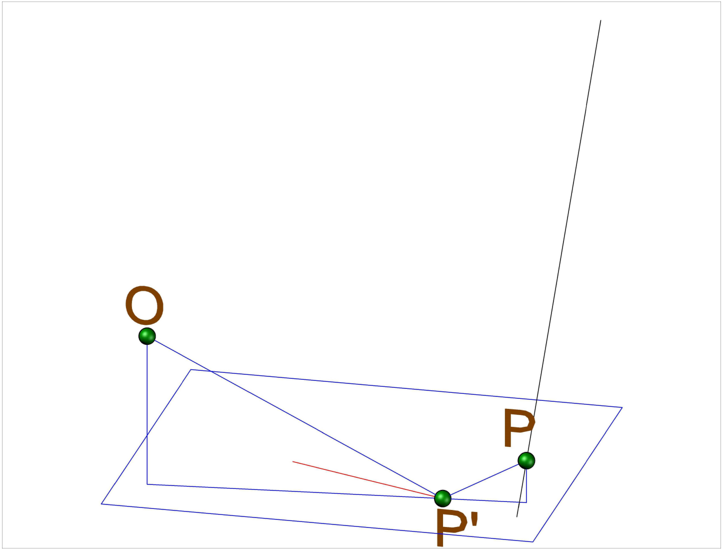

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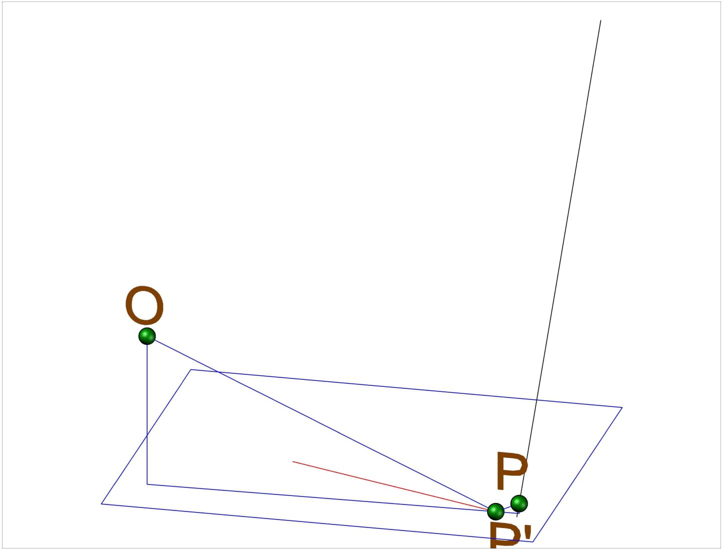
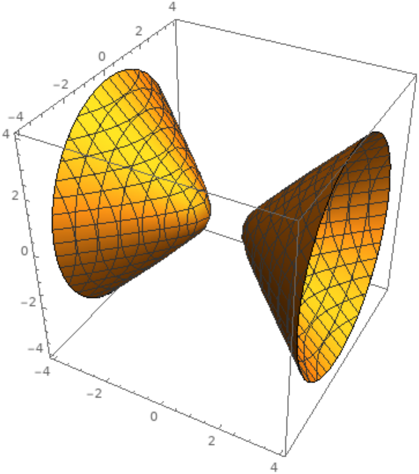
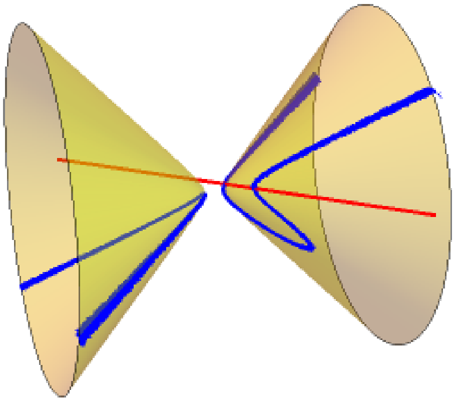


- Para um espelho hiperbolóide de duas folhas



$$f(X, Y, Z) = 3X^2 - 4Y^2 - 3Z^2 - 1, \quad O = (0, -1/2, -3/4)$$

Fotografias de rectas



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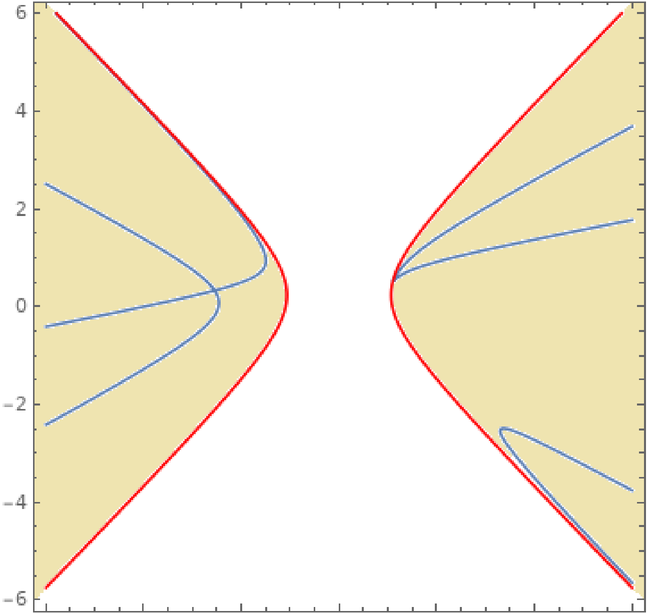
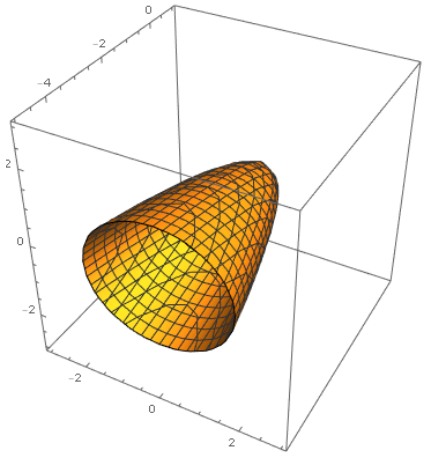


$$219298896 * X^6 - 1070347392 * X^5 * Y - 1341601296 * X^4 * Y^2 + 4375232640 * X^3 * Y^3 + 1899184384 * X^2 * Y^4 - 1950833664 * X * Y^5 + 498364416 * Y^6 + 1106009568 * X^5 - 1244703264 * X^4 * Y - 3492996000 * X^3 * Y^2 - 3102580672 * X^2 * Y^3 - 458388032 * X * Y^4 - 125641728 * Y^5 + 221502288 * X^4 + 3325975392 * X^3 * Y + 4890094512 * X^2 * Y^2 - 595553536 * X * Y^3 + 1546869136 * Y^4 - 2732273304 * X^3 - 213471472 * X^2 * Y - 1692868704 * X * Y^2 - 1774615168 * Y^3 + 366435436 * X^2 + 951468416 * X * Y + 1984199016 * Y^2 + 199241932 * X - 1928447200 * Y + 653312209 = 0$$

Fotografias de rectas

- Para um espelho parabolóide elíptico



$$f(X, Y, Z) = 3X^2 + 2Y + 4Z^2, \quad O = (-5, 4, -6)$$

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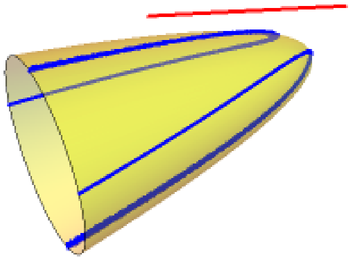
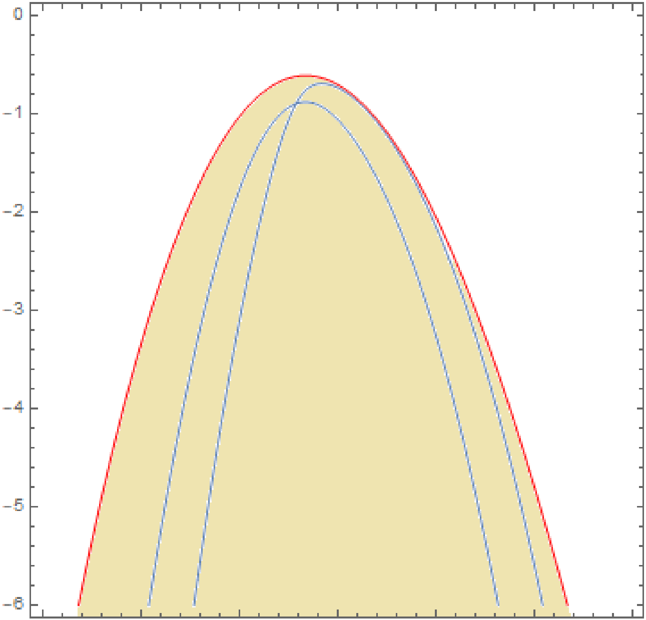
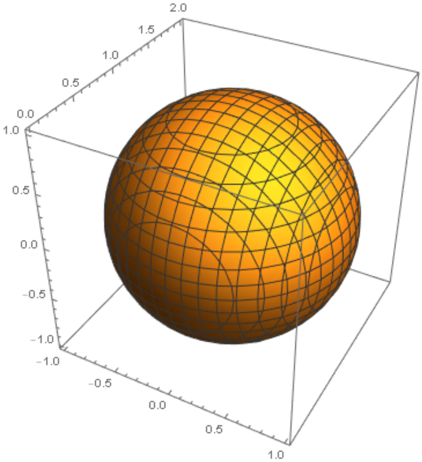


Fotografias de rectas

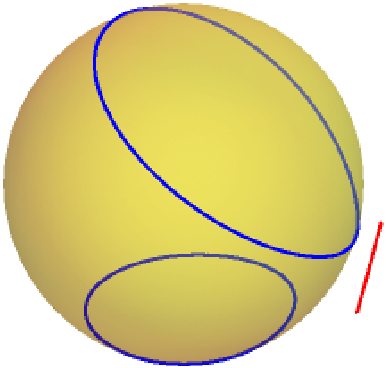


$$1010467908 * X^6 + 4816258164 * X^5 * Y + 11387142621 * X^4 * Y^2 + 695676612 * X^3 * Y^3 + 13124376 * X^2 * Y^4 + 70368 * X * Y^5 - 5383827972 * X^5 - 41585819526 * X^4 * Y - 43675276980 * X^3 * Y^2 + 11376798660 * X^2 * Y^3 + 386898576 * X * Y^4 + 1665376 * Y^5 + 31571107353 * X^4 + 111177090324 * X^3 * Y + 48938255478 * X^2 * Y^2 - 25275528516 * X * Y^3 + 3563987128 * Y^4 - 60416232132 * X^3 - 123010186248 * X^2 * Y - 17913650916 * X * Y^2 + 12598282236 * Y^3 + 64980727578 * X^2 + 73892242296 * X * Y - 2191951723 * Y^2 - 35409787248 * X - 18576225906 * Y + 9005421713 = 0$$

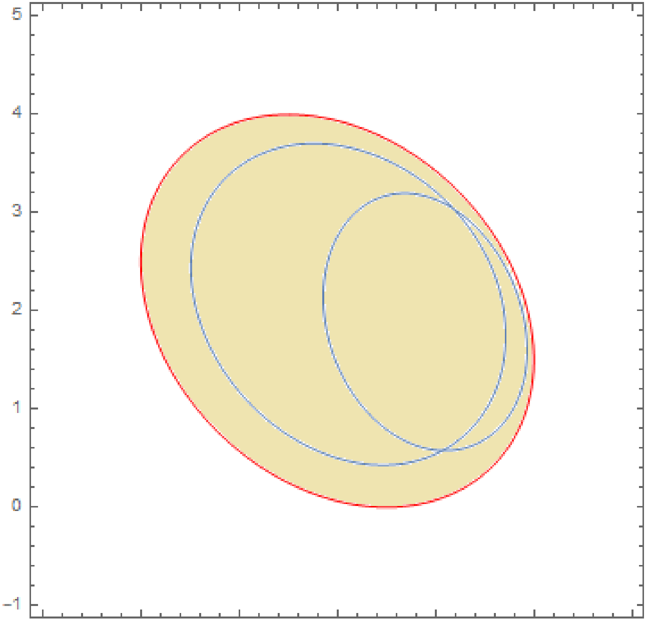
- Para um espelho esférico



$$f(X, Y, Z) = X^2 + (Y-1)^2 + Z^2 - 1, \quad O = (1, 0, -2)$$



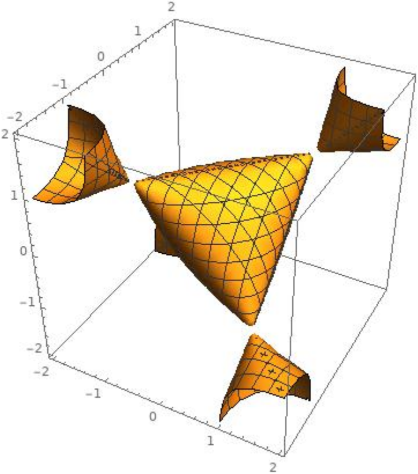
Fotografias de rectas



$$1604 * X^4 + 1188 * X^3 * Y + 2741 * X^2 * Y^2 + 876 * X * Y^3 + 953 * Y^4 + 820 * X^3 - 9258 * X^2 * Y - 2902 * X * Y^2 - 7122 * Y^3 + 3945 * X^2 + 192 * X * Y + 16703 * Y^2 + 730 * X - 12696 * Y + 3026 = 0$$

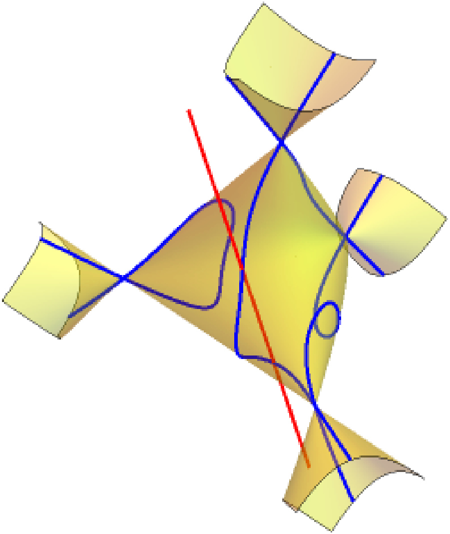
Grau da superfície do espelho	Grau máx. da curva na fotografia
2	6

- Para um espelho cúbico de Steinitz

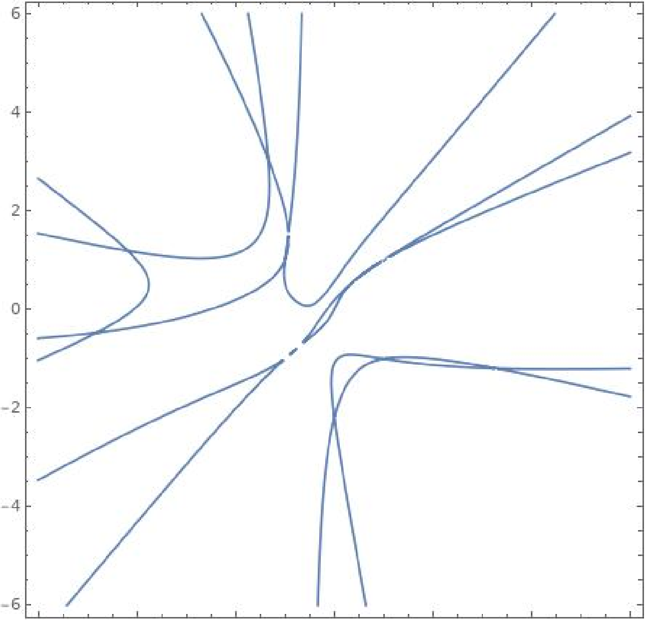


$$f(X,Y,Z) = X^2 + Y^2 + Z^2 - 2XYZ - 1, \quad O = (0,0,0)$$

Fotografias de rectas



Fotografias de rectas

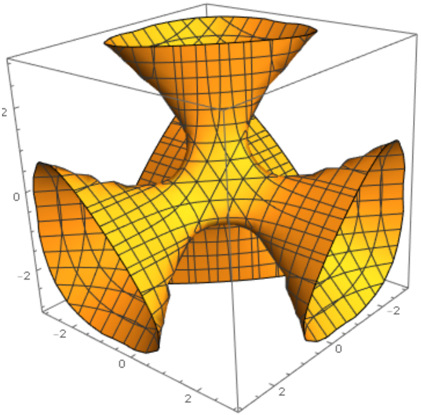


Fotografias de rectas

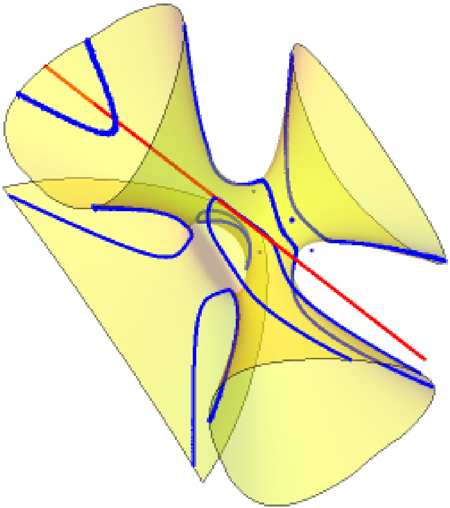
$$X^{14} * Y + X^{13} * Y^2 - 11 * X^{12} * Y^3 + 3 * X^{11} * Y^4 - 7 * X^{10} * Y^5 +$$
$$11 * X^9 * Y^6 + 27 * X^8 * Y^7 + 9 * X^7 * Y^8 + 31 * X^6 * Y^9 - 7 * X^5 * Y^{10} -$$
$$5 * X^4 * Y^{11} - 7 * X^3 * Y^{12} - 12 * X^2 * Y^{13} - 2 * X * Y^{14} +$$
$$X^{14} + 8 * X^{13} * Y - 2 * X^{12} * Y^2 - 36 * X^{11} * Y^3 + 37 * X^{10} * Y^4 -$$
$$138 * X^9 * Y^5 + 145 * X^8 * Y^6 - 104 * X^7 * Y^7 + 165 * X^6 * Y^8 - 30 * X^5 * Y^9 +$$
$$61 * X^4 * Y^{10} - 10 * X * Y^{13} - Y^{14} + 9 * X^{13} + 36 * X^{12} * Y -$$
$$59 * X^{11} * Y^2 - 67 * X^{10} * Y^3 + 2 * X^9 * Y^4 + 92 * X^8 * Y^5 + 147 * X^7 * Y^6 -$$
$$885 * X^6 * Y^7 + 442 * X^5 * Y^8 + 126 * X^4 * Y^9 + 177 * X^3 * Y^{10} - 37 * X^2 * Y^{11} -$$
$$17 * X * Y^{12} - 4 * Y^{13} + 35 * X^{12} + 96 * X^{11} * Y - 177 * X^{10} * Y^2 -$$
$$138 * X^9 * Y^3 - 76 * X^8 * Y^4 + 1512 * X^7 * Y^5 - 1457 * X^6 * Y^6 - 630 * X^5 * Y^7 +$$
$$220 * X^4 * Y^8 + 204 * X^3 * Y^9 + 115 * X^2 * Y^{10} - 30 * X * Y^{11} - 7 * Y^{12} +$$
$$95 * X^{11} + 119 * X^{10} * Y - 474 * X^9 * Y^2 - 748 * X^8 * Y^3 + 1606 * X^7 * Y^4 +$$
$$2777 * X^6 * Y^5 - 4398 * X^5 * Y^6 + 380 * X^4 * Y^7 + 428 * X^3 * Y^8 + 112 * X^2 * Y^9 -$$
$$29 * X * Y^{10} - 15 * Y^{11} + 191 * X^{10} + 162 * X^9 * Y - 1076 * X^8 * Y^2 -$$
$$1740 * X^7 * Y^3 + 4869 * X^6 * Y^4 + 216 * X^5 * Y^5 - 2915 * X^4 * Y^6 + 384 * X^3 * Y^7 +$$
$$310 * X^2 * Y^8 - 23 * Y^{10} + 287 * X^9 + 111 * X^8 * Y - 2061 * X^7 * Y^2 -$$
$$727 * X^6 * Y^3 + 5842 * X^5 * Y^4 - 3351 * X^4 * Y^5 - 137 * X^3 * Y^6 + 399 * X^2 * Y^7 -$$
$$11 * X * Y^8 - 23 * Y^9 + 393 * X^8 + 28 * X^7 * Y - 2703 * X^6 * Y^2 +$$
$$1506 * X^5 * Y^3 + 2032 * X^4 * Y^4 - 1716 * X^3 * Y^5 + 237 * X^2 * Y^6 + 154 * X * Y^7 -$$
$$33 * Y^8 + 405 * X^7 - 109 * X^6 * Y - 1970 * X^5 * Y^2 + 2068 * X^4 * Y^3 -$$
$$740 * X^3 * Y^4 + 110 * X^2 * Y^5 + 45 * X * Y^6 - 15 * Y^7 + 377 * X^6 -$$
$$186 * X^5 * Y - 658 * X^4 * Y^2 + 528 * X^3 * Y^3 - 240 * X^2 * Y^4 + 120 * X * Y^5 -$$
$$13 * Y^6 + 277 * X^5 - 217 * X^4 * Y - 47 * X^3 * Y^2 - 63 * X^2 * Y^3 +$$
$$67 * X * Y^4 - 5 * Y^5 + 165 * X^4 - 132 * X^3 * Y - 11 * X^2 * Y^2 -$$
$$6 * X * Y^3 + 11 * Y^4 + 77 * X^3 - 82 * X^2 * Y + 27 * X * Y^2 -$$
$$5 * Y^3 + 22 * X^2 - 22 * X * Y + 6 * Y^2 + 2 * X - Y = 0$$

Fotografias de rectas

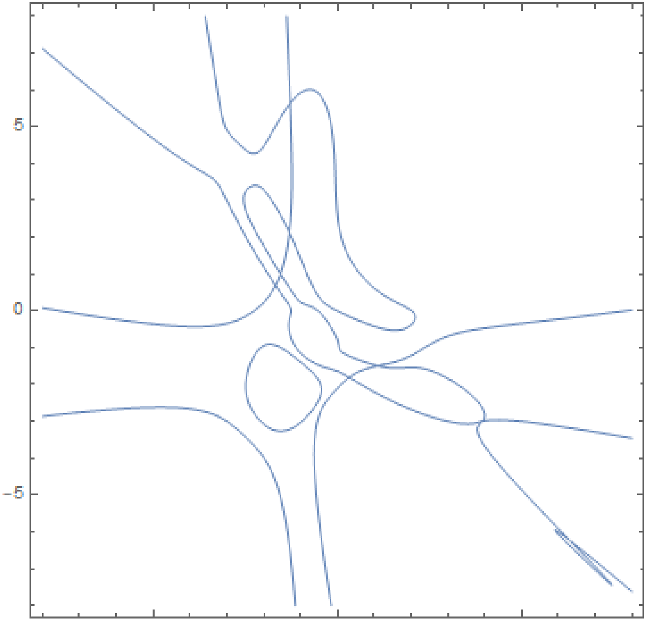
- Para um espelho cúbico de Clebsch



$$f(X,Y,Z) = X^3+Y^3+Z^3+1-(X+Y+Z+1)^3, \quad O=(0,0,-2)$$



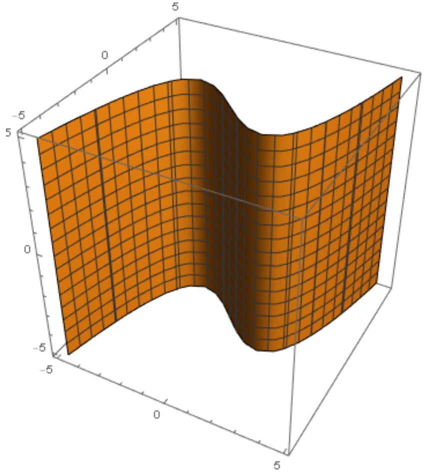
Fotografias de rectas



Fotografias de rectas

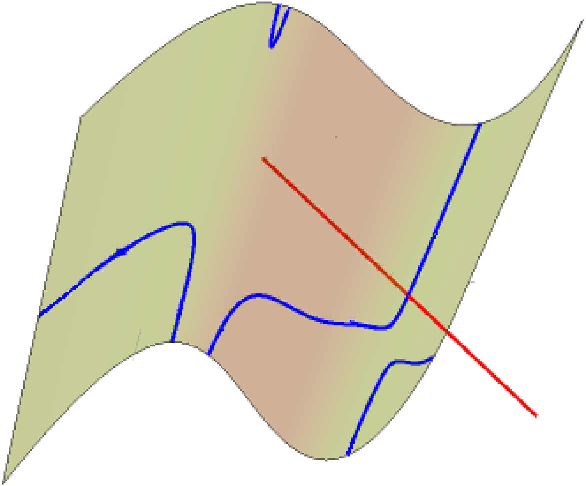
$$\begin{gathered}
28050 \ast X^{15} + 103200 \ast X^{14} \ast Y - 852300 \ast X^{13} \ast Y^{2} - 5061200 \ast X^{12} \ast Y^{3} - 10625897 \ast X^{11} \ast Y^{4} - \\
10225792 \ast X^{10} \ast Y^{5} - 6941174 \ast X^{9} \ast Y^{6} - 15255840 \ast X^{8} \ast Y^{7} - 29123953 \ast X^{7} \ast Y^{8} - 27593450 \ast X^{6} \ast Y^{9} - \\
13707268 \ast X^{5} \ast Y^{10} - 3930336 \ast X^{4} \ast Y^{11} - 1029502 \ast X^{3} \ast Y^{12} - 335742 \ast X^{2} \ast Y^{13} - 48348 \ast X \ast Y^{14} - \\
1360 \ast Y^{15} + 105540 \ast X^{14} - 2571720 \ast X^{13} \ast Y - 19419384 \ast X^{12} \ast Y^{2} - 53379876 \ast X^{11} \ast Y^{3} - \\
61590822 \ast X^{10} \ast Y^{4} - 28082868 \ast X^{9} \ast Y^{5} - 74967894 \ast X^{8} \ast Y^{6} - 247659924 \ast X^{7} \ast Y^{7} - 337523526 \ast X^{6} \ast Y^{8} - \\
232006968 \ast X^{5} \ast Y^{9} - 85282194 \ast X^{4} \ast Y^{10} - 17283984 \ast X^{3} \ast Y^{11} - 3056976 \ast X^{2} \ast Y^{12} - 806148 \ast X \ast Y^{13} - \\
58536 \ast Y^{14} - 1737324 \ast X^{13} - 22061664 \ast X^{12} \ast Y - 83028744 \ast X^{11} \ast Y^{2} - 84954636 \ast X^{10} \ast Y^{3} + \\
123588450 \ast X^{9} \ast Y^{4} + 127818738 \ast X^{8} \ast Y^{5} - 698721534 \ast X^{7} \ast Y^{6} - 1663121574 \ast X^{6} \ast Y^{7} - \\
1574542926 \ast X^{5} \ast Y^{8} - 757359738 \ast X^{4} \ast Y^{9} - 182448054 \ast X^{3} \ast Y^{10} - 18103662 \ast X^{2} \ast Y^{11} - 1280340 \ast X \ast Y^{12} - \\
402336 \ast Y^{13} - 7516584 \ast X^{12} - 39267180 \ast X^{11} \ast Y + 48348144 \ast X^{10} \ast Y^{2} + 727935552 \ast X^{9} \ast Y^{3} + \\
1384367220 \ast X^{8} \ast Y^{4} - 547171416 \ast X^{7} \ast Y^{5} - 4752954828 \ast X^{6} \ast Y^{6} - 6382717164 \ast X^{5} \ast Y^{7} - \\
4000777488 \ast X^{4} \ast Y^{8} - 1248070464 \ast X^{3} \ast Y^{9} - 156915792 \ast X^{2} \ast Y^{10} + 4914432 \ast X \ast Y^{11} + 1058400 \ast Y^{12} + \\
2045331 \ast X^{11} + 139547448 \ast X^{10} \ast Y + 1168120602 \ast X^{9} \ast Y^{2} + 2982040920 \ast X^{8} \ast Y^{3} + 410062338 \ast X^{7} \ast Y^{4} - \\
10245347208 \ast X^{6} \ast Y^{5} - 18816103818 \ast X^{5} \ast Y^{6} - 15120047412 \ast X^{4} \ast Y^{7} - 6069055653 \ast X^{3} \ast Y^{8} - \\
1113228036 \ast X^{2} \ast Y^{9} - 43249140 \ast X \ast Y^{10} + 9116064 \ast Y^{11} + 49675518 \ast X^{10} + 680070492 \ast X^{9} \ast Y + \\
2498082282 \ast X^{8} \ast Y^{2} - 338953896 \ast X^{7} \ast Y^{3} - 18803731716 \ast X^{6} \ast Y^{4} - 42271249680 \ast X^{5} \ast Y^{5} - \\
42540980652 \ast X^{4} \ast Y^{6} - 21703538196 \ast X^{3} \ast Y^{7} - 5347237356 \ast X^{2} \ast Y^{8} - 511051356 \ast X \ast Y^{9} + 717336 \ast Y^{10} + \\
88023834 \ast X^{9} + 630831402 \ast X^{8} \ast Y - 2807141346 \ast X^{7} \ast Y^{2} - 25919560008 \ast X^{6} \ast Y^{3} - 67587568506 \ast X^{5} \ast Y^{4} - \\
84311567712 \ast X^{4} \ast Y^{5} - 54532286586 \ast X^{3} \ast Y^{6} - 17472349782 \ast X^{2} \ast Y^{7} - 2318074200 \ast X \ast Y^{8} - \\
80977320 \ast Y^{9} - 33076188 \ast X^{8} - 2716927596 \ast X^{7} \ast Y - 22008222774 \ast X^{6} \ast Y^{2} - 69171704460 \ast X^{5} \ast Y^{3} - \\
107670349008 \ast X^{4} \ast Y^{4} - 88511275476 \ast X^{3} \ast Y^{5} - 36953480934 \ast X^{2} \ast Y^{6} - 6545131128 \ast X \ast Y^{7} - \\
278063928 \ast Y^{8} - 744890013 \ast X^{7} - 9728650800 \ast X^{6} \ast Y - 39860738766 \ast X^{5} \ast Y^{2} - 77189141484 \ast X^{4} \ast Y^{3} - \\
79263158877 \ast X^{3} \ast Y^{4} - 42940682118 \ast X^{2} \ast Y^{5} - 10534892724 \ast X \ast Y^{6} - 641140920 \ast Y^{7} - 1759896396 \ast X^{6} - \\
10005105096 \ast X^{5} \ast Y - 18989489178 \ast X^{4} \ast Y^{2} - 15188347584 \ast X^{3} \ast Y^{3} - 6606992610 \ast X^{2} \ast Y^{4} - \\
2221265916 \ast X \ast Y^{5} - 227063088 \ast Y^{6} - 72039780 \ast X^{5} + 9352062522 \ast X^{4} \ast Y + 38992534758 \ast X^{3} \ast Y^{2} + \\
52440235920 \ast X^{2} \ast Y^{3} + 26177130288 \ast X \ast Y^{4} + 3946362768 \ast Y^{5} + 5210956152 \ast X^{4} + \\
33714617040 \ast X^{3} \ast Y + 68415942870 \ast X^{2} \ast Y^{2} + 50537913336 \ast X \ast Y^{3} + 11423855736 \ast Y^{4} + 7781359122 \ast X^{3} + \\
33100271244 \ast X^{2} \ast Y + 40744518588 \ast X \ast Y^{2} + 14025790872 \ast Y^{3} + 4531065966 \ast X^{2} + \\
13704800508 \ast X \ast Y + 8213952096 \ast Y^{2} + 937461924 \ast X + 1874923848 \ast Y = 0
\end{gathered}$$

- Para um espelho cúbico cilíndrico

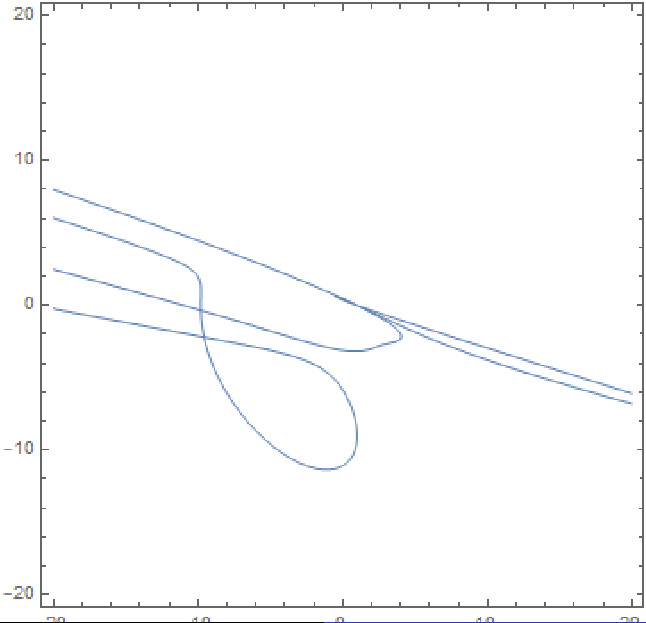


$$f(X, Y, Z) = X^3 - 10X - 10Y, \quad O = (1, 0, -2)$$

Fotografias de rectas



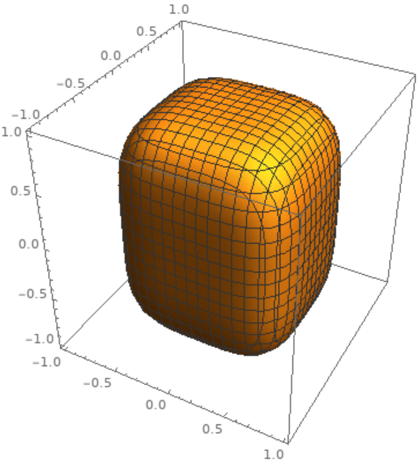
Fotografias de rectas



$$8000*X^8+132000*X^7*Y+879600*X^6*Y^2+3040200*X^5*Y^3+$$
$$5917992*X^4*Y^4+6882984*X^3*Y^5+5178466*X^2*Y^6+$$
$$2286183*X*Y^7+537120*Y^8+296000*X^7+3770400*X^6*Y+$$
$$19327680*X^5*Y^2+52113960*X^4*Y^3+81128064*X^3*Y^4+$$
$$73809324*X^2*Y^5+36690685*X*Y^6+9088377*Y^7+$$
$$2790800*X^6+26710200*X^5*Y+101977200*X^4*Y^2+$$
$$196162380*X^3*Y^3+191444130*X^2*Y^4+92179341*X*Y^5+$$
$$32352769*Y^6-383200*X^5-13287000*X^4*Y-$$
$$93934860*X^3*Y^2-243990900*X^2*Y^3-222143649*X*Y^4-$$
$$34653249*Y^5-50866000*X^4-293982000*X^3*Y-$$
$$536529420*X^2*Y^2-324162180*X*Y^3-56346537*Y^4+$$
$$140672000*X^3+635238000*X^2*Y+845038620*X*Y^2+$$
$$316836540*Y^3-161290000*X^2-492479400*X*Y-$$
$$336758820*Y^2+87063200*X+133897800*Y-18290800=0$$

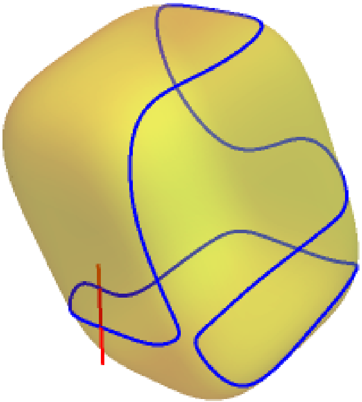
Grau da superfície do espelho	Grau máx. da curva na fotografia
3	15

- Para um espelho de ordem 4

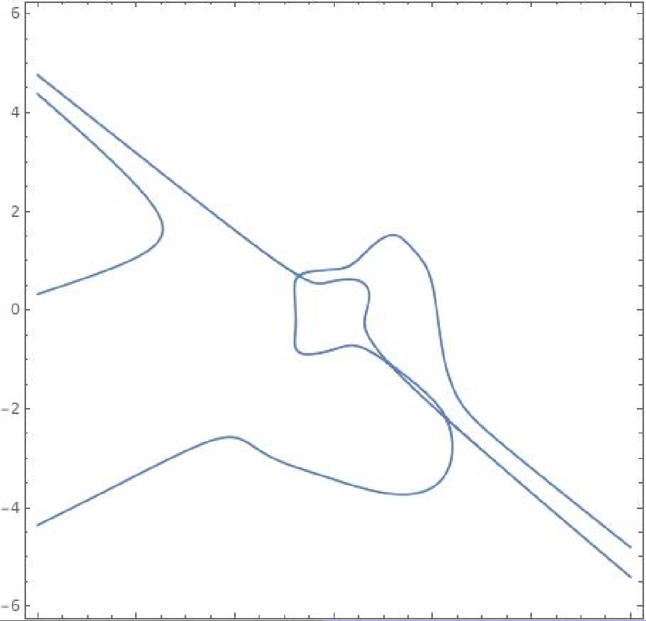


$$f(X, Y, Z) = 3X^4 + 4Y^4 + Z^4 - 1, \quad O = (0, 0, 0)$$

Fotografias de rectas



Fotografias de rectas



Fotografias de rectas

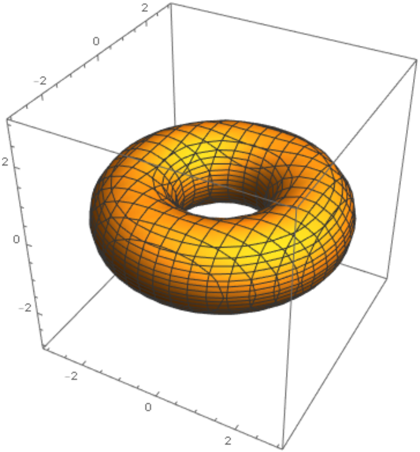
$$104976 * X^{28} - 629856 * X^{27} * Y + 1417176 * X^{26} * Y^{2} + 262440 * X^{25} * Y^{3} - 6466959 * X^{24} * Y^{4} +$$
$$6298560 * X^{23} * Y^{5} + 14206752 * X^{22} * Y^{6} - 16096320 * X^{21} * Y^{7} - 31446144 * X^{20} * Y^{8} +$$
$$44789760 * X^{19} * Y^{9} + 71326656 * X^{18} * Y^{10} - 78382080 * X^{17} * Y^{11} - 91473408 * X^{16} * Y^{12} +$$
$$127650816 * X^{15} * Y^{13} + 182476800 * X^{14} * Y^{14} - 86593536 * X^{13} * Y^{15} - 145635840 * X^{12} * Y^{16} +$$
$$139345920 * X^{11} * Y^{17} + 164192256 * X^{10} * Y^{18} - 53526528 * X^{9} * Y^{19} - 97026048 * X^{8} * Y^{20} +$$
$$75497472 * X^{7} * Y^{21} + 68665344 * X^{6} * Y^{22} - 35389440 * X^{5} * Y^{23} - 13762560 * X^{4} * Y^{24} +$$
$$14942208 * X^{3} * Y^{25} + 14155776 * X^{2} * Y^{26} - 14155776 * X * Y^{27} + 5308416 * Y^{28} + 1049760 * X^{27} -$$
$$4723920 * X^{26} * Y + 7085880 * X^{25} * Y^{2} + 15772644 * X^{24} * Y^{3} - 29393280 * X^{23} * Y^{4} - 23724576 * X^{22} * Y^{5} +$$
$$71523648 * X^{21} * Y^{6} + 71103744 * X^{20} * Y^{7} - 127464192 * X^{19} * Y^{8} - 99626112 * X^{18} * Y^{9} +$$
$$242362368 * X^{17} * Y^{10} + 263139840 * X^{16} * Y^{11} - 123393024 * X^{15} * Y^{12} - 207028224 * X^{14} * Y^{13} +$$
$$159252480 * X^{13} * Y^{14} + 286654464 * X^{12} * Y^{15} + 147750912 * X^{11} * Y^{16} + 663552 * X^{10} * Y^{17} +$$
$$12238848 * X^{9} * Y^{18} + 58392576 * X^{8} * Y^{19} + 195231744 * X^{7} * Y^{20} + 99385344 * X^{6} * Y^{21} -$$
$$89653248 * X^{5} * Y^{22} + 17694720 * X^{4} * Y^{23} + 62128128 * X^{3} * Y^{24} + 47185920 * X^{2} * Y^{25} -$$
$$70778880 * X * Y^{26} + 35389440 * Y^{27} + 3936600 * X^{26} - 11809800 * X^{25} * Y - 21375738 * X^{24} * Y^{2} +$$
$$31492800 * X^{23} * Y^{3} + 54377568 * X^{22} * Y^{4} - 85240512 * X^{21} * Y^{5} - 111694464 * X^{20} * Y^{6} +$$
$$227308032 * X^{19} * Y^{7} + 326592000 * X^{18} * Y^{8} - 254555136 * X^{17} * Y^{9} - 359188992 * X^{16} * Y^{10} +$$
$$517321728 * X^{15} * Y^{11} + 491028480 * X^{14} * Y^{12} - 295612416 * X^{13} * Y^{13} - 59968512 * X^{12} * Y^{14} +$$
$$664879104 * X^{11} * Y^{15} - 23224320 * X^{10} * Y^{16} - 537477120 * X^{9} * Y^{17} + 411402240 * X^{8} * Y^{18} +$$
$$605159424 * X^{7} * Y^{19} - 194641920 * X^{6} * Y^{20} - 403439616 * X^{5} * Y^{21} + 298450944 * X^{4} * Y^{22} +$$
$$176947200 * X^{3} * Y^{23} - 61341696 * X^{2} * Y^{24} - 117964800 * X * Y^{25} + 88473600 * Y^{26} + 5861160 * X^{25} +$$
$$53773956 * X^{24} * Y - 4723920 * X^{23} * Y^{2} - 62653176 * X^{22} * Y^{3} + 195908544 * X^{21} * Y^{4} +$$
$$252922176 * X^{20} * Y^{5} - 234446400 * X^{19} * Y^{6} - 254298528 * X^{18} * Y^{7} + 734614272 * X^{17} * Y^{8} +$$
$$304560000 * X^{16} * Y^{9} - 912619008 * X^{15} * Y^{10} + 16422912 * X^{14} * Y^{11} + 951395328 * X^{13} * Y^{12} -$$
$$539011584 * X^{12} * Y^{13} - 1280765952 * X^{11} * Y^{14} + 639498240 * X^{10} * Y^{15} + 450330624 * X^{9} * Y^{16} -$$
$$949764096 * X^{8} * Y^{17} - 543277056 * X^{7} * Y^{18} + 345710592 * X^{6} * Y^{19} - 211746816 * X^{5} * Y^{20} -$$
$$315064320 * X^{4} * Y^{21} + 159514624 * X^{3} * Y^{22} - 23592960 * X^{2} * Y^{23} - 164364288 * X * Y^{24} + 80609280 * Y^{25} -$$
$$46217871 * X^{24} + 14486688 * X^{23} * Y + 10235160 * X^{22} * Y^{2} - 186822288 * X^{21} * Y^{3} - 218070144 * X^{20} * Y^{4} +$$
$$49688640 * X^{19} * Y^{5} - 123031872 * X^{18} * Y^{6} - 822265344 * X^{17} * Y^{7} - 495144576 * X^{16} * Y^{8} -$$
$$75209472 * X^{15} * Y^{9} - 553734144 * X^{14} * Y^{10} - 1111532544 * X^{13} * Y^{11} - 675012096 * X^{12} * Y^{12} -$$
$$607150080 * X^{11} * Y^{13} - 671791104 * X^{10} * Y^{14} - 395145216 * X^{9} * Y^{15} - 754089984 * X^{8} * Y^{16} -$$
$$903757824 * X^{7} * Y^{17} - 200196096 * X^{6} * Y^{18} + 166330368 * X^{5} * Y^{19} - 448757760 * X^{4} * Y^{20} -$$

Fotografias de rectas

$$
\begin{gathered}
369426432 * X^{3} * Y^{21} + 25165824 * X^{2} * Y^{22} + 96731136 * X * Y^{23} - 104005632 * Y^{24} - 11022480 * X^{23} - \\
47081736 * X^{22} * Y - 35586864 * X^{21} * Y^{2} + 55847232 * X^{20} * Y^{3} - 132829632 * X^{19} * Y^{4} - \\
321506496 * X^{18} * Y^{5} + 28180224 * X^{17} * Y^{6} + 417011328 * X^{16} * Y^{7} - 344445696 * X^{15} * Y^{8} - \\
759932928 * X^{14} * Y^{9} + 405015552 * X^{13} * Y^{10} + 788299776 * X^{12} * Y^{11} - 397357056 * X^{11} * Y^{12} - \\
572479488 * X^{10} * Y^{13} + 780005376 * X^{9} * Y^{14} + 396804096 * X^{8} * Y^{15} - 189480960 * X^{7} * Y^{16} + \\
134922240 * X^{6} * Y^{17} + 602210304 * X^{5} * Y^{18} - 165445632 * X^{4} * Y^{19} - 115408896 * X^{3} * Y^{20} + \\
174587904 * X^{2} * Y^{21} + 128188416 * X * Y^{22} - 94371840 * Y^{23} + 56482920 * X^{22} + 27328752 * X^{21} * Y - \\
29655720 * X^{20} * Y^{2} + 201378960 * X^{19} * Y^{3} + 414821412 * X^{18} * Y^{4} + 93265344 * X^{17} * Y^{5} + \\
15761952 * X^{16} * Y^{6} + 960289344 * X^{15} * Y^{7} + 945271296 * X^{14} * Y^{8} + 95530752 * X^{13} * Y^{9} + \\
461045952 * X^{12} * Y^{10} + 1748293632 * X^{11} * Y^{11} + 793082880 * X^{10} * Y^{12} + 5142528 * X^{9} * Y^{13} + \\
1032652800 * X^{8} * Y^{14} + 1275899904 * X^{7} * Y^{15} + 70106112 * X^{6} * Y^{16} + 2211840 * X^{5} * Y^{17} + \\
708026368 * X^{4} * Y^{18} + 379797504 * X^{3} * Y^{19} - 71958528 * X^{2} * Y^{20} + 13959168 * X * Y^{21} + 94453760 * Y^{22} + \\
11255760 * X^{21} + 72748368 * X^{20} * Y + 23724576 * X^{19} * Y^{2} - 76020120 * X^{18} * Y^{3} + 205752960 * X^{17} * Y^{4} + \\
244220832 * X^{16} * Y^{5} - 145986624 * X^{15} * Y^{6} - 163233792 * X^{14} * Y^{7} + 458618112 * X^{13} * Y^{8} + \\
175509504 * X^{12} * Y^{9} - 576073728 * X^{11} * Y^{10} - 18662400 * X^{10} * Y^{11} + 195581952 * X^{9} * Y^{12} - \\
227045376 * X^{8} * Y^{13} - 400637952 * X^{7} * Y^{14} + 70723584 * X^{6} * Y^{15} - 248020992 * X^{5} * Y^{16} - \\
148267008 * X^{4} * Y^{17} + 54771712 * X^{3} * Y^{18} - 87293952 * X^{2} * Y^{19} - 130744320 * X * Y^{20} + 87490560 * Y^{21} - \\
73115784 * X^{20} - 28028592 * X^{19} * Y + 7348320 * X^{18} * Y^{2} - 250682688 * X^{17} * Y^{3} - 352276128 * X^{16} * Y^{4} - \\
185270976 * X^{15} * Y^{5} - 207650304 * X^{14} * Y^{6} - 783385344 * X^{13} * Y^{7} - 699912576 * X^{12} * Y^{8} - \\
497166336 * X^{11} * Y^{9} - 552379392 * X^{10} * Y^{10} - 808704000 * X^{9} * Y^{11} - 771692544 * X^{8} * Y^{12} - \\
670519296 * X^{7} * Y^{13} - 366796800 * X^{6} * Y^{14} - 296828928 * X^{5} * Y^{15} - 476553216 * X^{4} * Y^{16} - \\
368492544 * X^{3} * Y^{17} - 15335424 * X^{2} * Y^{18} - 12386304 * X * Y^{19} - 116686848 * Y^{20} - 11975040 * X^{19} - \\
60052104 * X^{18} * Y - 45314640 * X^{17} * Y^{2} + 62692056 * X^{16} * Y^{3} - 82005696 * X^{15} * Y^{4} - \\
239842944 * X^{14} * Y^{5} + 12764736 * X^{13} * Y^{6} + 239969952 * X^{12} * Y^{7} - 180361728 * X^{11} * Y^{8} - \\
302807808 * X^{10} * Y^{9} + 317039616 * X^{9} * Y^{10} + 247670784 * X^{8} * Y^{11} - 171657216 * X^{7} * Y^{12} - \\
18220032 * X^{6} * Y^{13} + 390131712 * X^{5} * Y^{14} - 9529344 * X^{4} * Y^{15} - 57212928 * X^{3} * Y^{16} + \\
121208832 * X^{2} * Y^{17} + 74072064 * X * Y^{18} - 80732160 * Y^{19} + 66240828 * X^{18} + 48638880 * X^{17} * Y - \\
5581224 * X^{16} * Y^{2} + 193805136 * X^{15} * Y^{3} + 336234240 * X^{14} * Y^{4} + 153669312 * X^{13} * Y^{5} + \\
151345152 * X^{12} * Y^{6} + 666620928 * X^{11} * Y^{7} + 505771776 * X^{10} * Y^{8} + 205493760 * X^{9} * Y^{9} + \\
494014464 * X^{8} * Y^{10} + 722774016 * X^{7} * Y^{11} + 250463232 * X^{6} * Y^{12} + 140894208 * X^{5} * Y^{13} + \\
438958080 * X^{4} * Y^{14} + 246435840 * X^{3} * Y^{15} + 3145728 * X^{2} * Y^{16} + 73580544 * X * Y^{17} + 82640896 * Y^{18} +
\end{gathered}
$$

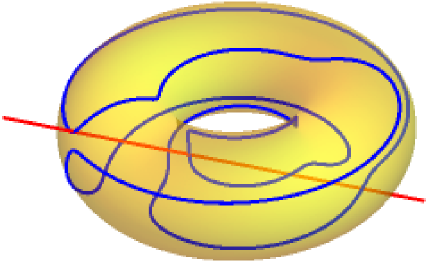
$$
\begin{gathered}
8573040 \ast X^{17} + 40223304 \ast X^{16} \ast Y + 27398736 \ast X^{15} \ast Y^{2} - 34805376 \ast X^{14} \ast Y^{3} + 65271744 \ast X^{13} \ast Y^{4} + \\
95551488 \ast X^{12} \ast Y^{5} - 25754112 \ast X^{11} \ast Y^{6} - 47464704 \ast X^{10} \ast Y^{7} + 35043840 \ast X^{9} \ast Y^{8} + 37103616 \ast X^{8} \ast Y^{9} - \\
91330560 \ast X^{7} \ast Y^{10} - 42633216 \ast X^{6} \ast Y^{11} - 79810560 \ast X^{5} \ast Y^{12} + 3631104 \ast X^{4} \ast Y^{13} - 15790080 \ast X^{3} \ast Y^{14} - \\
55148544 \ast X^{2} \ast Y^{15} - 36913152 \ast X \ast Y^{16} + 41287680 \ast Y^{17} - 45793512 \ast X^{16} - 41115600 \ast X^{15} \ast Y - \\
7371648 \ast X^{14} \ast Y^{2} - 121424832 \ast X^{13} \ast Y^{3} - 193038714 \ast X^{12} \ast Y^{4} - 144960192 \ast X^{11} \ast Y^{5} - \\
117209376 \ast X^{10} \ast Y^{6} - 276924096 \ast X^{9} \ast Y^{7} - 295256448 \ast X^{8} \ast Y^{8} - 210249216 \ast X^{7} \ast Y^{9} - \\
167414208 \ast X^{6} \ast Y^{10} - 197268480 \ast X^{5} \ast Y^{11} - 201670144 \ast X^{4} \ast Y^{12} - 125826048 \ast X^{3} \ast Y^{13} - \\
35450880 \ast X^{2} \ast Y^{14} - 56279040 \ast X \ast Y^{15} - 52492800 \ast Y^{16} - 3998160 \ast X^{15} - 22394880 \ast X^{14} \ast Y - \\
18790704 \ast X^{13} \ast Y^{2} + 21925728 \ast X^{12} \ast Y^{3} - 21679488 \ast X^{11} \ast Y^{4} - 53942112 \ast X^{10} \ast Y^{5} + 19376064 \ast X^{9} \ast Y^{6} + \\
37013760 \ast X^{8} \ast Y^{7} - 29355264 \ast X^{7} \ast Y^{8} - 10774656 \ast X^{6} \ast Y^{9} + 76658688 \ast X^{5} \ast Y^{10} - 1230336 \ast X^{4} \ast Y^{11} - \\
4674560 \ast X^{3} \ast Y^{12} + 33546240 \ast X^{2} \ast Y^{13} + 20447232 \ast X \ast Y^{14} - 16005120 \ast Y^{15} + 27006048 \ast X^{14} + \\
26873856 \ast X^{13} \ast Y + 6639732 \ast X^{12} \ast Y^{2} + 70217280 \ast X^{11} \ast Y^{3} + 98602272 \ast X^{10} \ast Y^{4} + 64400832 \ast X^{9} \ast Y^{5} + \\
73941120 \ast X^{8} \ast Y^{6} + 150211584 \ast X^{7} \ast Y^{7} + 92606976 \ast X^{6} \ast Y^{8} + 61185024 \ast X^{5} \ast Y^{9} + 110034432 \ast X^{4} \ast Y^{10} + \\
69755904 \ast X^{3} \ast Y^{11} + 18616320 \ast X^{2} \ast Y^{12} + 33030144 \ast X \ast Y^{13} + 25715712 \ast Y^{14} + 2101680 \ast X^{13} + \\
9243072 \ast X^{12} \ast Y + 9016272 \ast X^{11} \ast Y^{2} - 7307496 \ast X^{10} \ast Y^{3} + 2942784 \ast X^{9} \ast Y^{4} + 14074560 \ast X^{8} \ast Y^{5} - \\
3038400 \ast X^{7} \ast Y^{6} - 11333088 \ast X^{6} \ast Y^{7} - 11255040 \ast X^{5} \ast Y^{8} + 3904896 \ast X^{4} \ast Y^{9} - 8054272 \ast X^{3} \ast Y^{10} - \\
14063616 \ast X^{2} \ast Y^{11} - 6441984 \ast X \ast Y^{12} + 6551040 \ast Y^{13} - 12284730 \ast X^{12} - 15497568 \ast X^{11} \ast Y - \\
5103000 \ast X^{10} \ast Y^{2} - 26452656 \ast X^{9} \ast Y^{3} - 41427072 \ast X^{8} \ast Y^{4} - 32208192 \ast X^{7} \ast Y^{5} - 24411456 \ast X^{6} \ast Y^{6} - \\
39301632 \ast X^{5} \ast Y^{7} - 39394944 \ast X^{4} \ast Y^{8} - 20143872 \ast X^{3} \ast Y^{9} - 12386304 \ast X^{2} \ast Y^{10} - 16634880 \ast X \ast Y^{11} - \\
8451584 \ast Y^{12} - 952560 \ast X^{11} - 2995704 \ast X^{10} \ast Y - 3044304 \ast X^{9} \ast Y^{2} + 1819584 \ast X^{8} \ast Y^{3} - \\
1664064 \ast X^{7} \ast Y^{4} - 2348352 \ast X^{6} \ast Y^{5} + 4108032 \ast X^{5} \ast Y^{6} - 1634688 \ast X^{4} \ast Y^{7} + 1006848 \ast X^{3} \ast Y^{8} + \\
4036608 \ast X^{2} \ast Y^{9} + 2411520 \ast X \ast Y^{10} - 2580480 \ast Y^{11} + 4534776 \ast X^{10} + 6111504 \ast X^{9} \ast Y + \\
2910168 \ast X^{8} \ast Y^{2} + 10063728 \ast X^{7} \ast Y^{3} + 11322084 \ast X^{6} \ast Y^{4} + 9218880 \ast X^{5} \ast Y^{5} + 10702816 \ast X^{4} \ast Y^{6} + \\
8778432 \ast X^{3} \ast Y^{7} + 5248512 \ast X^{2} \ast Y^{8} + 5339904 \ast X \ast Y^{9} + 3209536 \ast Y^{10} + 212400 \ast X^{9} + 843696 \ast X^{8} \ast Y + \\
956448 \ast X^{7} \ast Y^{2} - 589032 \ast X^{6} \ast Y^{3} - 565632 \ast X^{5} \ast Y^{4} + 654048 \ast X^{4} \ast Y^{5} - 789184 \ast X^{3} \ast Y^{6} - \\
1225728 \ast X^{2} \ast Y^{7} - 374016 \ast X \ast Y^{8} + 449280 \ast Y^{9} - 1275912 \ast X^{8} - 2142288 \ast X^{7} \ast Y - 987552 \ast X^{6} \ast Y^{2} - \\
2414016 \ast X^{5} \ast Y^{3} - 3006432 \ast X^{4} \ast Y^{4} - 1733184 \ast X^{3} \ast Y^{5} - 1534464 \ast X^{2} \ast Y^{6} - 1721088 \ast X \ast Y^{7} - \\
603264 \ast Y^{8} - 125760 \ast X^{7} - 79704 \ast X^{6} \ast Y - 159408 \ast X^{5} \ast Y^{2} - 179640 \ast X^{4} \ast Y^{3} + 133568 \ast X^{3} \ast Y^{4} + \\
153216 \ast X^{2} \ast Y^{5} + 78528 \ast X \ast Y^{6} - 276000 \ast Y^{7} + 254748 \ast X^{6} + 517536 \ast X^{5} \ast Y + \\
345384 \ast X^{4} \ast Y^{2} + 379248 \ast X^{3} \ast Y^{3} + 448512 \ast X^{2} \ast Y^{4} + 318528 \ast X \ast Y^{5} + 130944 \ast Y^{6} + 15120 \ast X^{5} + \\
18360 \ast X^{4} \ast Y - 6480 \ast X^{3} \ast Y^{2} - 5760 \ast X^{2} \ast Y^{3} + 6720 \ast X \ast Y^{4} + 23040 \ast Y^{5} - 12920 \ast X^{4} - 66000 \ast X^{3} \ast Y - \\
55080 \ast X^{2} \ast Y^{2} - 38760 \ast X \ast Y^{3} - 5615 \ast Y^{4} - 5840 \ast X^{3} + 720 \ast X^{2} \ast Y + 1080 \ast X \ast Y^{2} - \\
11460 \ast Y^{3} + 600 \ast X^{2} + 1800 \ast X \ast Y + 1350 \ast Y^{2} + 1000 \ast X + 1500 \ast Y + 625 = 0
\end{gathered}
$$

- Para um espelho tórico

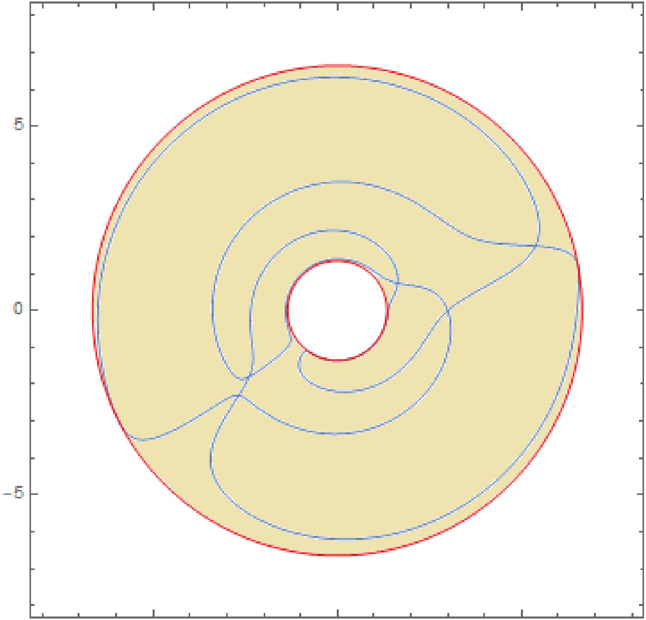


$$f(X, Y, Z) = (X^2 + Y^2 + Z^2 + 3)^2 - 16(X^2 + Y^2), \quad O = (0, 0, -2)$$

Fotografias de rectas



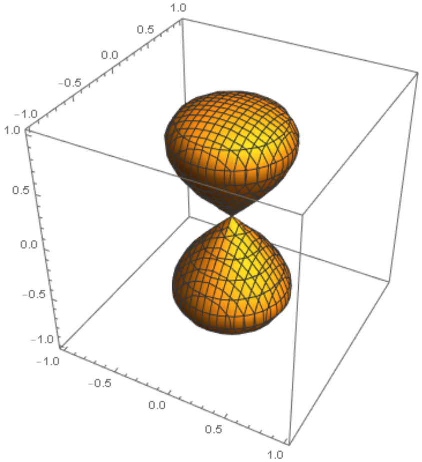
Fotografias de rectas



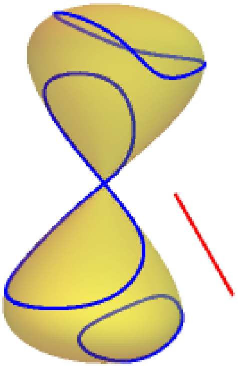
Fotografias de rectas

$$
\begin{aligned}
&33361 \ast X^{12} - 275484 \ast X^{11} \ast Y + 1023178 \ast X^{10} \ast Y^{2} - 2271180 \ast X^{9} \ast Y^{3} + 4292271 \ast X^{8} \ast Y^{4} - \\
&6329880 \ast X^{7} \ast Y^{5} + 7604524 \ast X^{6} \ast Y^{6} - 8117400 \ast X^{5} \ast Y^{7} + 6791311 \ast X^{4} \ast Y^{8} - \\
&4952460 \ast X^{3} \ast Y^{9} + 3022410 \ast X^{2} \ast Y^{10} - 1169244 \ast X \ast Y^{11} + 533169 \ast Y^{12} - \\
&8316 \ast X^{11} - 26196 \ast X^{10} \ast Y + 134748 \ast X^{9} \ast Y^{2} - 266028 \ast X^{8} \ast Y^{3} + \\
&622152 \ast X^{7} \ast Y^{4} - 802152 \ast X^{6} \ast Y^{5} + 974808 \ast X^{5} \ast Y^{6} - 1072248 \ast X^{4} \ast Y^{7} + \\
&663732 \ast X^{3} \ast Y^{8} - 671172 \ast X^{2} \ast Y^{9} + 168012 \ast X \ast Y^{10} - 161244 \ast Y^{11} - \\
&2058696 \ast X^{10} + 16579764 \ast X^{9} \ast Y - 59010336 \ast X^{8} \ast Y^{2} + 118296720 \ast X^{7} \ast Y^{3} - \\
&195394680 \ast X^{6} \ast Y^{4} + 255411576 \ast X^{5} \ast Y^{5} - 252708984 \ast X^{4} \ast Y^{6} + 222252048 \ast X^{3} \ast Y^{7} - \\
&144981792 \ast X^{2} \ast Y^{8} + 68557428 \ast X \ast Y^{9} - 30715848 \ast Y^{10} + 479628 \ast X^{9} + \\
&721980 \ast X^{8} \ast Y - 4610304 \ast X^{7} \ast Y^{2} + 7799328 \ast X^{6} \ast Y^{3} - 16708680 \ast X^{5} \ast Y^{4} + \\
&19066104 \ast X^{4} \ast Y^{5} - 17667936 \ast X^{3} \ast Y^{6} + 17622144 \ast X^{2} \ast Y^{7} - 6049188 \ast X \ast Y^{8} + \\
&5633388 \ast Y^{9} + 30930174 \ast X^{8} - 239081220 \ast X^{7} \ast Y + 815204412 \ast X^{6} \ast Y^{2} - \\
&1449219276 \ast X^{5} \ast Y^{3} + 1968577020 \ast X^{4} \ast Y^{4} - 2181194892 \ast X^{3} \ast Y^{5} + 1615261500 \ast X^{2} \ast Y^{6} - \\
&971056836 \ast X \ast Y^{7} + 430958718 \ast Y^{8} - 5955444 \ast X^{7} + 5488884 \ast X^{6} \ast Y + \\
&21454956 \ast X^{5} \ast Y^{2} - 27722412 \ast X^{4} \ast Y^{3} + 60776244 \ast X^{3} \ast Y^{4} - 71911476 \ast X^{2} \ast Y^{5} + \\
&33365844 \ast X \ast Y^{6} - 38700180 \ast Y^{7} - 153760680 \ast X^{6} + 1131276780 \ast X^{5} \ast Y - \\
&3605336568 \ast X^{4} \ast Y^{2} + 5474959128 \ast X^{3} \ast Y^{3} - 5334687864 \ast X^{2} \ast Y^{4} + 4343682348 \ast X \ast Y^{5} - \\
&1883111976 \ast Y^{6} + 8984196 \ast X^{5} + 7654500 \ast X^{4} \ast Y - 81548856 \ast X^{3} \ast Y^{2} + \\
&88792200 \ast X^{2} \ast Y^{3} - 90533052 \ast X \ast Y^{4} + 81137700 \ast Y^{5} + 195891777 \ast X^{4} - \\
&1384423488 \ast X^{3} \ast Y + 4165250850 \ast X^{2} \ast Y^{2} - 5193792576 \ast X \ast Y^{3} + 2230956513 \ast Y^{4} = 0
\end{aligned}
$$

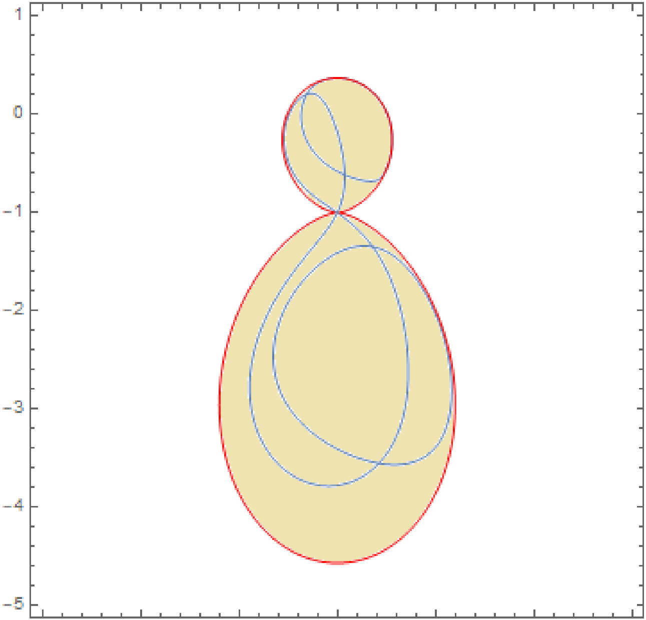
- Para um espelho de revolução da lemniscata de Gerono



$$f(X, Y, Z) = X^2 + Y^2 - Z^2 + Z^4, \quad O = (0, 2, -2)$$



Fotografias de rectas



Fotografias de rectas

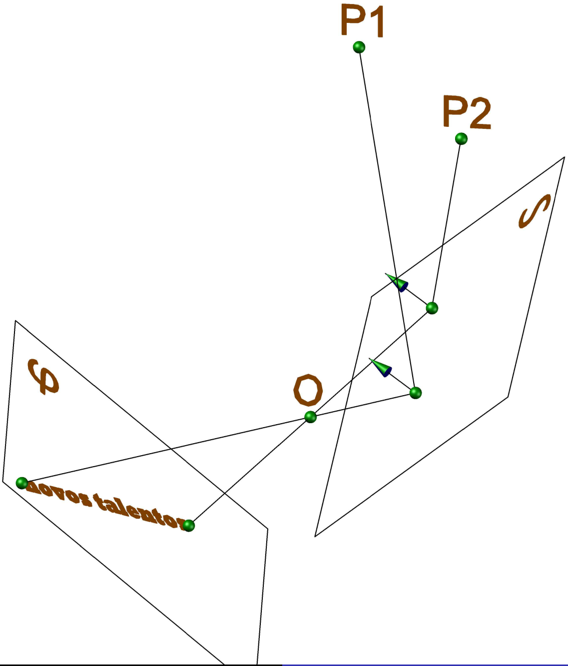
$$81936 * X^{18} - 331968 * X^{17} * Y + 717776 * X^{16} * Y^{2} - 414976 * X^{15} * Y^{3} + 4054800 * X^{14} * Y^{4} +$$
$$4998080 * X^{13} * Y^{5} + 13907152 * X^{12} * Y^{6} + 20112768 * X^{11} * Y^{7} + 28163632 * X^{10} * Y^{8} + 34306240 * X^{9} * Y^{9} +$$
$$34208496 * X^{8} * Y^{10} + 31992832 * X^{7} * Y^{11} + 24735536 * X^{6} * Y^{12} + 16878144 * X^{5} * Y^{13} + 9987440 * X^{4} * Y^{14} +$$
$$4676480 * X^{3} * Y^{15} + 1848192 * X^{2} * Y^{16} + 516608 * X * Y^{17} + 63232 * Y^{18} - 6989376 * X^{17} +$$
$$16563136 * X^{16} * Y + 8311296 * X^{15} * Y^{2} + 138023808 * X^{14} * Y^{3} + 204626752 * X^{13} * Y^{4} + 479020608 * X^{12} * Y^{5} +$$
$$665527680 * X^{11} * Y^{6} + 905831936 * X^{10} * Y^{7} + 1038388800 * X^{9} * Y^{8} + 1015264320 * X^{8} * Y^{9} +$$
$$908513024 * X^{7} * Y^{10} + 679852416 * X^{6} * Y^{11} + 451321536 * X^{5} * Y^{12} + 255713728 * X^{4} * Y^{13} +$$
$$116687232 * X^{3} * Y^{14} + 44229888 * X^{2} * Y^{15} + 11691520 * X * Y^{16} + 1376256 * Y^{17} + 187141280 * X^{16} +$$
$$13724928 * X^{15} * Y + 1206775584 * X^{14} * Y^{2} + 1007945792 * X^{13} * Y^{3} + 3997778304 * X^{12} * Y^{4} +$$
$$4991800896 * X^{11} * Y^{5} + 8393144704 * X^{10} * Y^{6} + 10385706624 * X^{9} * Y^{7} + 11204922528 * X^{8} * Y^{8} +$$
$$11303360896 * X^{7} * Y^{9} + 9086575392 * X^{6} * Y^{10} + 6682888512 * X^{5} * Y^{11} + 4097294912 * X^{4} * Y^{12} +$$
$$1991116608 * X^{3} * Y^{13} + 825201216 * X^{2} * Y^{14} + 223462400 * X * Y^{15} + 24559872 * Y^{16} - 1478684608 * X^{15} +$$
$$1705612800 * X^{14} * Y - 5950636864 * X^{13} * Y^{2} + 8262538880 * X^{12} * Y^{3} - 2656440576 * X^{11} * Y^{4} +$$
$$20358510848 * X^{10} * Y^{5} + 20502416256 * X^{9} * Y^{6} + 33218590464 * X^{8} * Y^{7} + 41066683968 * X^{7} * Y^{8} +$$
$$34571293184 * X^{6} * Y^{9} + 31828324800 * X^{5} * Y^{10} + 19867230848 * X^{4} * Y^{11} + 10395436672 * X^{3} * Y^{12} +$$
$$4738333440 * X^{2} * Y^{13} + 946891264 * X * Y^{14} + 25390080 * Y^{15} + 8598983208 * X^{14} - 11889117920 * X^{13} * Y +$$
$$44446902264 * X^{12} * Y^{2} - 36081377280 * X^{11} * Y^{3} + 85393162672 * X^{10} * Y^{4} - 10984149696 * X^{9} * Y^{5} +$$
$$82526358960 * X^{8} * Y^{6} + 61625141376 * X^{7} * Y^{7} + 50063275080 * X^{6} * Y^{8} + 67969035552 * X^{5} * Y^{9} +$$
$$22651658584 * X^{4} * Y^{10} + 16198227072 * X^{3} * Y^{11} + 4741645536 * X^{2} * Y^{12} - 3353776768 * X * Y^{13} -$$
$$827853312 * Y^{14} - 28217438416 * X^{13} + 74987435712 * X^{12} * Y - 155733688080 * X^{11} * Y^{2} +$$
$$278112222464 * X^{10} * Y^{3} - 242030064480 * X^{9} * Y^{4} + 328344247872 * X^{8} * Y^{5} - 76780329504 * X^{7} * Y^{6} +$$
$$82376419392 * X^{6} * Y^{7} + 61404143472 * X^{5} * Y^{8} - 85344997888 * X^{4} * Y^{9} - 3966282192 * X^{3} * Y^{10} -$$
$$43752246336 * X^{2} * Y^{11} - 27636940352 * X * Y^{12} - 1250290176 * Y^{13} + 75262502569 * X^{12} -$$
$$223913850480 * X^{11} * Y + 585060004102 * X^{10} * Y^{2} - 768926925540 * X^{9} * Y^{3} + 1125911446344 * X^{8} * Y^{4} -$$
$$781000975212 * X^{7} * Y^{5} + 590687449230 * X^{6} * Y^{6} - 177375736908 * X^{5} * Y^{7} - 206926934577 * X^{4} * Y^{8} +$$
$$42464654924 * X^{3} * Y^{9} - 162902826516 * X^{2} * Y^{10} - 16319736928 * X * Y^{11} + 18569694144 * Y^{12} -$$
$$142257672792 * X^{11} + 604181891960 * X^{10} * Y - 1247650707084 * X^{9} * Y^{2} + 2223113610672 * X^{8} * Y^{3} -$$
$$2306202812148 * X^{7} * Y^{4} + 2300417670456 * X^{6} * Y^{5} - 1152057383604 * X^{5} * Y^{6} + 487886293248 * X^{4} * Y^{7} +$$
$$307251026172 * X^{3} * Y^{8} - 111031433280 * X^{2} * Y^{9} + 259003242736 * X * Y^{10} + 83257033152 * Y^{11} +$$

Fotografias de rectas

$$260628788014 \ast X^{10} - 1051701815244 \ast X^{9} \ast Y + 2539496480472 \ast X^{8} \ast Y^{2} - 3613652947392 \ast X^{7} \ast Y^{3} + 4708710286290 \ast X^{6} \ast Y^{4} - 3013975587852 \ast X^{5} \ast Y^{5} + 2874758880036 \ast X^{4} \ast Y^{6} + 366960188088 \ast X^{3} \ast Y^{7} + 561498825708 \ast X^{2} \ast Y^{8} + 822542752976 \ast X \ast Y^{9} + 115967600880 \ast Y^{10} - 371451634724 \ast X^{9} + 1642275724536 \ast X^{8} \ast Y - 3253461192800 \ast X^{7} \ast Y^{2} + 5603198210672 \ast X^{6} \ast Y^{3} - 4450766827956 \ast X^{5} \ast Y^{4} + 5645787467816 \ast X^{4} \ast Y^{5} - 611851469176 \ast X^{3} \ast Y^{6} + 1664991379248 \ast X^{2} \ast Y^{7} + 949807284992 \ast X \ast Y^{8} - 36677702656 \ast Y^{9} + 475962333468 \ast X^{8} - 1671455042356 \ast X^{7} \ast Y + 3938761142130 \ast X^{6} \ast Y^{2} - 3906304206612 \ast X^{5} \ast Y^{3} + 5867547373294 \ast X^{4} \ast Y^{4} - 2072363776128 \ast X^{3} \ast Y^{5} + 2120970974904 \ast X^{2} \ast Y^{6} + 137053245472 \ast X \ast Y^{7} - 253808215872 \ast Y^{8} - 402346983884 \ast X^{7} + 1560356515224 \ast X^{6} \ast Y - 1963669113324 \ast X^{5} \ast Y^{2} + 3256764939728 \ast X^{4} \ast Y^{3} - 2126430284016 \ast X^{3} \ast Y^{4} + 1503902951856 \ast X^{2} \ast Y^{5} - 541962171328 \ast X \ast Y^{6} - 142260181824 \ast Y^{7} + 286936457270 \ast X^{6} - 491420458356 \ast X^{5} \ast Y + 687133610276 \ast X^{4} \ast Y^{2} - 608467989752 \ast X^{3} \ast Y^{3} + 708155737752 \ast X^{2} \ast Y^{4} - 57734818880 \ast X \ast Y^{5} + 271102097952 \ast Y^{6} - 43026165852 \ast X^{5} - 152939921432 \ast X^{4} \ast Y + 548048907144 \ast X^{3} \ast Y^{2} + 346606197264 \ast X^{2} \ast Y^{3} + 641309245280 \ast X \ast Y^{4} + 461267722176 \ast Y^{5} - 76689033901 \ast X^{4} + 496176590004 \ast X^{3} \ast Y + 171293398812 \ast X^{2} \ast Y^{2} + 504095066944 \ast X \ast Y^{3} + 262969579200 \ast Y^{4} + 119399665460 \ast X^{3} + 36502893456 \ast X^{2} \ast Y + 57243136976 \ast X \ast Y^{2} + 29611869312 \ast Y^{3} - 2582500068 \ast X^{2} - 67871923472 \ast X \ast Y - 31363926288 \ast Y^{2} - 20087592736 \ast X - 12617648832 \ast Y - 1276032320 = 0$$

Grau da superfície do espelho	Grau máx. da curva na fotografia
4	28

Grau da superfície do espelho	Grau máx. da curva na fotografia
1	1
2	6
3	15
4	28
Conjectura	
n	n(2n-1)


P1
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