Fixed point theory and periodic solutions of differential equations with non-invertible linear part

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04/09/2020

Brouwer Fixed Point Theorem

Let B be a closed ball in $\mathbb{R}^n$ and $f : B \to B$ a continuous mapping. Then there exists $x^* \in B$ such that $f(x^*) = x^*$.

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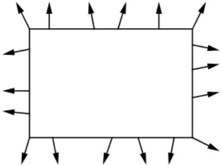
Poincaré-Miranda Theorem

Let $L \in \mathbb{R}^n$, $L_i \geq 0$, $\Omega = \{x \in \mathbb{R}^n : |x_i| \leq L_i, i = 1, ..., n\}$ and $f : \Omega \to \mathbb{R}^n$ a continuous vector field such that

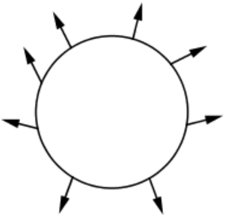
$$f_i(x_1, \ldots, x_{i-1}, -L_i, x_{i+1}, \ldots, x_n) \geq 0, \text{ for all } 1 \leq i \leq n$$
$$f_i(x_1, \ldots, x_{i-1}, L_i, x_{i+1}, \ldots, x_n) \leq 0, \text{ for all } 1 \leq i \leq n$$

Then there is a point $x^* \in \Omega$ such that $f(x^*) = 0$.



Theorem 1

Let $\mathbb{B}^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. If $f : \mathbb{B}^n \to \mathbb{R}^n$ is a continuous vector field such that $\forall x \in \partial\mathbb{B}^n, f(x) \cdot x > 0$, then the equation $f(x) = 0$ has a solution in $\mathbb{B}^n$.



Periodic solution of

$$u''(t) + \lambda u(t) + F(u(t))' = e(t)$$

F is class C^1 and bounded in $\mathbb{R}$ and F' is bounded in $\mathbb{R}$.

e is a 2π-periodic continuous function.

λ a real parameter.

Considering the homogeneous linear part $u'' + \lambda u = 0$ it is particularly interesting to consider the case where λ is an eigenvalue, i.e., there are non-trivial 2π-periodic solutions.

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By integration we get

$$\begin{cases} u'(t) + F(u(t)) = E(t) + C \\ u(0) = u(2\pi), u'(0) = u'(2\pi) \end{cases}$$

Where C is a constant and E is a fixed antiderivative.

Note that E is 2π-periodic because $\int_0^{2\pi} e(t)dt = 0$.

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The solution of this problem will depend on x and C, therefore we have a solution

$u(t, x, C)$. By the integral formula of solutions we have that u is a solution of (1) if and only if

$$u(t, x, C) = x + \int_0^t (E(s) + C - F(u(s, x, C)))ds$$

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Theorem

Let $J = [a, b]$, $K \subset \mathbb{R}$ compact and $f : J \times \mathbb{R} \times K \to \mathbb{R}$ continuous and Lipchitz in the second variable variable. Let $\sigma = (\alpha, \beta, \mu)$. Then the integral equation

$$u(x, \sigma) = \beta + \int_\alpha^x f(t, u(t, \sigma), \mu)dt$$

has a unique solution $\forall \alpha \in J, \beta \in \mathbb{R}, \mu \in K$ and the solution $u(x, \sigma)$, is continuous in $J \times J \times \mathbb{R} \times K$.

Because we want a 2π-periodic solution

$$u(0) = u(2\pi) \iff 2\pi C + \int_0^{2\pi} E(s) - F(u(s,x,C))ds = 0$$

F and E are bounded functions (E is class C^1 in $[0, 2\pi]$) we can conclude that in order for u to be a periodic function, C has lower and upper bound, a and b, respectively.

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Let $h : [a, b] \to \mathbb{R}$, $h(C) = 2\pi C + \int_0^{2\pi} E(s) - F(u(s, x, C))ds$.

By considering a large enough b and a small enough a we have $h(a) < 0$ and $h(b) > 0$ and by applying Bolzano's theorem there is a $C' \in]a, b[$ such that $h(C') = 0$, that is $u(t, x, C')$ is 2π-periodic.

Main result and proofs

$$u''(t) + u(t) + F(u(t))' = e(t)$$

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Theorem

Let $e(t)$ be a 2π-periodic function, $F(\infty)$ and $F(-\infty)$ exist and be finite. By setting

$$e_c = \int_0^{2\pi} \cos(x)e(x)dx, \text{ and } e_s = \int_0^{2\pi} \sin(x)e(x)dx$$

Then

1. *The condition $2(F(\infty) - F(-\infty)) > \sqrt{e_s^2 + e_c^2}$ is sufficient to the existence of a 2π-periodic solution of the differential equation $u''(t) + u(t) + F(u(t))' = e(t)$.*
2. *If $\forall x \in \mathbb{R}, m \leq F(x) \leq M$, for some $m, M \in \mathbb{R}$ with $m \leq M$, then the condition $2(M - m) \geq \sqrt{e_s^2 + e_c^2}$ is necessary to the existence of a 2π-periodic solution of the differential equation $u''(t) + u(t) + F(u(t))' = e(t)$.*

We want to prove the existence of a solution to the problem

$$\begin{cases} u''(t) + u(t) + F(u(t))' = e(t) \\ u(0) = u(2\pi), u'(0) = u'(2\pi) \end{cases} \qquad (2)$$

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Let u be a class C^2 function. Consider the decomposition

$$u(x) = A\cos(x) + B\sin(x) + W(x, A, B)$$

where $A = u(0)$ and $u'(0) = B$.

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Therefore $W(0, A, B) = W'(0, A, B) = 0$.

$$\begin{cases} W''(t,A,B)+W(t,A,B)+F(A\cos(t) + B\sin(t) + W(t, A, B))' = e(t) \\ W(0,A,B) = W(2\pi, A, B) = 0, W'(0, A, B) = W'(2\pi, A, B) = 0 \end{cases} \quad (3)$$

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$$W(t, A, B) = \int_0^t \sin(t - x)(e(x) - F(A\cos(x) + B\sin(x) + W(x, A, B))')dx$$

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Using integration by parts we get the system

$$\begin{cases} e_s+\int_0^{2\pi} \cos(x)F(A\cos(x)+B\sin(x)+W(x,A,B))dx=0 \\ e_c-\int_0^{2\pi} \sin(x)F(A\cos(x)+B\sin(x)+W(x,A,B))dx=0 \end{cases}$$

We define the vector field $(X(A,B), Y(A,B))$:

$$\begin{cases} \mathrm{X(A,B)} = \mathrm{e}_c - \int_0^{2\pi} \sin(x) F(A\cos(x) + B\sin(x) + W(x,A,B))dx \\ Y(A,B) = e_s + \int_0^{2\pi} \cos(x) F(A\cos(x) + B\sin(x) + W(x,A,B))dx \end{cases}$$

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Let $(A,B) \in \mathbb{R}^2$, $R = \sqrt{A^2 + B^2}$ and $\phi \in \mathbb{R}$, such that

$$\cos(\phi) = \frac{B}{\sqrt{A^2+B^2}}, \ \sin(\phi) = \frac{A}{\sqrt{A^2+B^2}}.$$

Consider the dot product:

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$$\int_0^{2\pi} B\sin(x) F(A\cos(x) + B\sin(x) + W(x,A,B)) dx =$$
$$= \mathrm{B}e_c - Ae_s - \int_0^{2\pi} (A\cos(x) + B\sin(x)) F(A\cos(x) + B\sin(x) + W(x,A,B)) dx =$$

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$$= \text{Be}_c - Ae_s - \int_0^{2\pi} R\sin(x+\phi)F(R\sin(x+\phi) + W(x,A,B))dx$$
$$\leq \|(A,B)\| \times \|(e_s,e_c)\| - \int_0^{2\pi} R\sin(x+\phi)F(R\sin(x+\phi) + W(x,A,B))dx =$$
$$= \text{R}\times\sqrt{e_c^2 + e_s^2} - \int_0^{2\pi} R\sin(x+\phi)F(R\sin(x+\phi) + W(x,A,B))dx =$$
$$= \text{R}\left(\sqrt{e_c^2 + e_s^2} - \int_0^{2\pi} \sin(x+\phi)F(R\sin(x+\phi) + W(x,A,B))dx\right)$$

With the assumptions on F we have that

$$\sqrt{e_c^2 + e_s^2} - \int_0^{2\pi} \sin(x+\phi)F(R\sin(x+\phi) + W(x,A,B))dx \rightarrow \sqrt{e_c^2 + e_s^2} - 2(F(\infty) - F(-\infty)) < 0$$

as $R \to \infty$ uniformly with respect to $\phi \in \mathbb{R}$. Note that W is a bounded function in $[0, 2\pi]$ independently of A,B.

Main result and proofs

With the assumptions on F we have that

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Therefore there is $R_0 \in \mathbb{R}$ such that,

$$\forall (A,B) \in \mathbb{R}^2 : \|(A,B)\| \geq R_0 \implies (-Y(A,B), X(A,B)) \cdot (A,B) < 0$$

and in particular,

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By theorem 1 there is a $(A^*, B^*) \in \mathbb{R}^2$ with $\|(A^*, B^*)\| < R_0$ such that $X(A^*, B^*) = Y(A^*, B^*) = 0$.

Hence we have a solution of (2).

Final remarks

$$u''(t) + u(t) + F(u(t))' + g(u(t)) = e(t)$$

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