# Representations of the Higman-Thompson groups

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# Introduction

## Definition

The Thompson group $F$ is the group of piecewise linear homeomorphisms of the unit interval which are:

- differentiable everywhere except at a finite subset of $\left\{\frac{a}{2^b} : a, b \in \mathbb{Z}\right\}$ - the dyadic rationals;
- have a derivative equal to a power of 2, where it exists.

Objectives:

1. Show a generalisation of the Thompson group;
2. Define a family of representations of said generalizations;
3. Introduce a sufficient condition for two such representations to be equivalent;
4. Investigate when it might be possible to satisfy this condition.

# Transition matrices

Consider the following:

- System with $n$ different states $1, \ldots, n$;
- Only certain transitions are allowed;

## Example

The following matrix encodes the allowed transitions for a system with 4 different states.

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Entry $i, j$ of the matrix is equal to 1 if and only if the system can transition from state $i$ to state $j$.

# Transition matrices

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

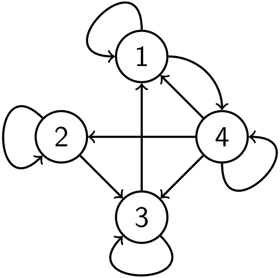


Figure: Transition matrix and corresponding graph.

How to encode the evolution of the system?

# Admissible words

## Example

Consider a system with at least two states, for which the $1 \to 2$ and $2 \to 1$ transitions are both allowed. The infinite word

$$12121212\ldots$$

describes a possible evolution of the system which indefinitely alternates between states 1 and 2.

## Definition

Let $A = [A(i,j)]_{n\times n}$ be a transition matrix. Define the set of admissible words, $X_A$ as

$$\{(x_m)_{m\in\mathbb{N}} \in \{1,\ldots,n\}^{\mathbb{N}} : \forall m \in \mathbb{N}, A(x_m, x_{m+1}) = 1\}$$

# Finite words as subsets of $X_A$

## Definition

Let $x = x_1x_2\dots x_m$ be a finite word. Then, let

$$x = \{y \in X_A : x \text{ is a prefix of } y\}$$

that is, $x$ is the set of elements in $X_A$ which have $x$ as a prefix. Furthermore, $xy$ denotes the concatenation of $x$ and $y \in X_A$.

## Example

Let $y = 121212\dots$ Then,

- $y \in 1$ because 1 is a prefix of $y$;
- $y \in 12$ because 12 is also a prefix of $y$;
- $3y$ is the word 3121212...

# Cylinder maps

## Definition

Consider two families of admissible finite words $\mu_1, \dots, \mu_r$ and $\nu_1, \dots, \nu_r$ such that:

- $X_A = \bigsqcup_{i=1}^r \mu_i = \bigsqcup_{i=1}^r \nu_i$;
- $\mu_i y \in X_A \Leftrightarrow \nu_i y \in X_A$ for $i \in \{1, \dots, r\}$.

The bijective map $\tau = (\mu_i x) \mapsto (\nu_i x) : X_A \to X_A$ is called a *cylinder* map.

## Example

Consider the transition matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and the following $\mu_i$ and $\nu_i$.

| $\mu_i$ | 11 | 121 | 2 |
|---|---|---|---|
| $\nu_i$ | 1 | 211 | 212 |

Then, for instance, $\tau(121212\dots) = 211212\dots$

# Perron-Frobenius theorem

## Theorem (Perron-Frobenius)

Let $A$ be an irreducible non-negative $n \times n$ matrix with spectral radius $\lambda$. Then, $\lambda$ is a simple eigenvalue of $A$ and the associated eigenvector is strictly positive.

## Example

The polynomial $t^4 - 4t^3 + 5t^2 - 3t$ is the characteristic polynomial of matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and it has a real simple root, $\lambda \approx 2.47$ of greatest absolute value, which is an eigenvalue of the matrix, with eigenvector $\left(\frac{1}{\lambda(\lambda-1)}, \frac{1}{\lambda(\lambda-1)^3}, \frac{1}{\lambda(\lambda-1)^2}, \frac{1}{\lambda}\right)$.

# $\lambda$-adic functions

## Proposition

There is $\rho_A : X_A \to [0, 1)$ surjective and order-preserving.

Cylinder map:

$$\begin{array}{ccccc} \mu_1 & \cdots & \mu_i & \cdots & \mu_r \\ \updownarrow & & \updownarrow & & \updownarrow \\ \nu_1 & \cdots & \nu_i & \cdots & \nu_r \end{array}$$

$\lambda$-adic function:

$$\begin{array}{ccccc} \rho_A(\mu_1) & \cdots & \rho_A(\mu_i) & \cdots & \rho_A(\mu_r) \\ \updownarrow & & \updownarrow & & \updownarrow \\ \rho_A(\nu_1) & \cdots & \rho_A(\nu_i) & \cdots & \rho_A(\mu_r) \end{array}$$

- piecewise linear mapping with positive slope;
- intervals of the form $[a, b)$ (so it's well-defined);
- slopes are powers of $\lambda$, singularities are a finite subset of $\mathbb{Z}\left[\frac{1}{\lambda}\right]$.

# The group $\Gamma_A$

**Definition**

The group $\Gamma_A$ is the group of $\lambda$-adic functions obtained from cylinder maps. $F_A$ is the subgroup of the continuous elements of $\Gamma_A$.

**Proposition**

$F_A$ is a subgroup of the piecewise linear homeomorphisms of $[0, 1)$ to $[0, 1)$ which are:

- differentiable everywhere except at a finite subset of $\mathbb{Z}\left[\frac{1}{\lambda}\right]$;
- have a derivative equal to a power of $\lambda$, where it exists.

# The shift operator $\sigma$

Take a bijective restriction of $\rho_A$.

**Definition**

The shift operator $\sigma : X_A \to X_A$ is such that $\sigma(x_1 x_2 x_3 \ldots) = x_2 x_3 \ldots$

**Definition**

Let $\text{orb}(x) = \{y \in [0,1) : \sigma^r(\rho_A^{-1}(x)) = \sigma^s(\rho_A^{-1}(y))$ for some $r, s \in \mathbb{N}\}$.

**Example**

- $\sigma(3121212\ldots) = 121212\ldots$
- $\sigma^2(21121212\ldots) = 121212\ldots$
- $\rho_A(3121212\ldots) \in \text{orb}(\rho_A(21121212\ldots))$

# The Hilbert space $H_x$

**Definition**

Let $H_x = \ell^2(\text{orb}(x))$, the Hilbert space of sequences $u : \text{orb}(x) \to \mathbb{C}$ such that $\sum_{w \in \text{orb}(x)} |u(w)|^2$ converges.

**Definition**

Let $y \in \text{orb}(x)$. Then $\delta_y \in H_x$ is such that

$$\delta_y(w) = \begin{cases} 1 & \text{if } y = w \\ 0 & \text{otherwise} \end{cases}$$

**Proposition**

The set $\{\delta_y : y \in \text{orb}(x)\}$ is an Hilbert basis for $H_x$.

# Representations of $F_A$

**Proposition**

Let $f \in \Gamma_A$ and $x \in [0,1)$. Then, $f(x) \in \mathrm{orb}(x)$.

**Proposition**

For $x \in [0,1)$, there is a representation $\tau_x : F_A \to B(H_x)$ such that $\tau_x(f)\delta_y = \delta_{f(y)}$ on the basis of $H_x$.

Family of representations of $F_A$: $\{\tau_x : x \in [0,1)\}$.

**Question**

When are $\tau_y$ and $\tau_z$, in some sense, the same representation?

Idea: if $f \in F_A$ is such that $f(x) = x \Leftrightarrow x \in \{0, y\}$, then $\tau_y$ and $\tau_z$ are equivalent if and only if $z \in \mathrm{orb}(y)$.

# Fixed points

## Proposition

Suppose $f(x) = x$ for $f \in F_A$ and $x \in [0,1)$. Then, the word $\rho_A^{-1}(x) \in X_A$ is eventually periodic.

## Question

Suppose $y \in X_A$ is eventually periodic. Is there always $f \in F_A$ such that $f(x) = x$ if and only if $x \in \{0, \rho_A(y)\}$? How to find out if $\rho_A^{-1}(x)$ is eventually periodic?

# $\beta$-transformations

Suppose $A$ is such that every row of $A$ but the last is filled with 1 and the last row is a series of 1 (at least one) followed by a series of 0.

**Proposition**

In the above conditions, $\rho_A^{-1}(x)$ is periodic if and only if $x$ has a periodic representation in base $\beta$ (the Perron-Frobenius eigenvalue).

**Definition**

Suppose $x > 1$ is a real algebraic integer with minimal polynomial $p(x)$ such that every $y \neq x$ with $p(y) = 0$ satisfies $|p(y)| < 1$. Then, $x$ is called a Pisot number.

**Proposition**

If $\beta$ is a Pisot number, $x$ has a periodic representation in base $\beta$ if and only if $x \in \mathbb{Q}[\beta]$.

# The Higman-Thompson groups

Suppose $A$ is an $n \times n$ matrix such that every entry is equal to 1. $A$ has Perron-Frobenius eigenvalue equal to $n$.
In this case, $F_A$ is simply the Higman-Thompson group $F_n$:

## Definition

The Higman-Thompson group $F_n$ is the group of piecewise linear homeomorphisms of the unit interval which are:

- differentiable everywhere except at a finite subset of $\left\{\frac{a}{n^b} : a, b \in \mathbb{Z}\right\}$ - the $n$-adic rationals;
- have a derivative equal to a power of $n$, where it exists.

## Proposition

Let $q \in \mathbb{Q} \cap [0,1) = \mathbb{Q}[n] \cap [0,1)$. There exists $f \in F_n$ such that $f(x) = x$ if and only if $x \in \{0, q\}$.