The Genus of a Laurent Polynomial

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Introduction and preliminaries

Riemann Surface

A Riemann surface is a complex manifold of dimension one, that is a set that around each point is locally homeomorphic to $\mathbb{C}$.

Definition - Holomorphic map

A holomorphic map between Riemann surfaces $F : X \to Y$ is one which is holomorphic when viewed as a map between regions in the complex plane.

Definition

Let $F : X \longrightarrow Y$ be a non-constant holomorphic map defined at $p \in X$. The multiplicity of F at p, denoted $mult_p(F)$, is the unique integer m such that there are local coordinates near p and $F(p)$ with F having the form $z \to z^m$. If $m > 1$ we call p a ramification point and $F(p)$ a branch point

Affine plane curves

Definition

An affine plane curve is the locus of zeros in $\mathbb{C}^2$ of a polynomial $f(x, y)$. A polynomial $f(x, y)$ is nonsingular at a root p if either partial derivative $\frac{\partial f}{\partial x}$ or $\frac{\partial f}{\partial y}$ is non zero at p. The affine plane curve X of roots of f is nonsingular at p if f is nonsingualar at p. The curve X is nonsingular, or smooth, if it is non singular at each of its points.

Theorem

If $f \in \mathbb{C}[x, y]$ is an irreducible polynomial then its locus of roots, X, is connected. Hence if f is a nonsingular and irreducible, X is a Riemann surface.

Lemma

Let X be a smooth afine plane curve defined by $f(x, y) = 0$, where f is a polynomial. Define $\pi : X \longrightarrow \mathbb{C}$ by $\pi(x, y) = x$, then π is ramified at $p \in X$ if and only if $\frac{\partial f}{\partial y}(p) = 0$.

Riemann-Hurwitz formula

Proposition

Let $F : X \longrightarrow Y$ be a non-constant holomorphic map between compact Riemann surfaces. For each $y \in Y$, define

$$d_y(F) = \sum_{p \in F^{-1}(y)} mult_p(F)$$

Then $d_y(F)$ is constant, independently of $y \in Y$.
We call this constant the degree of F and we denote it by $deg(F)$.

Riemann-Hurwitz Formula

Let X, Y be two compact Riemann surfaces and $F : X \longrightarrow Y$ be a non-constant holomorphic map. Then

$$2g(X) - 2 = deg(F)(2g(Y) - 2) + \sum_{p \in X} [mult_p(F) - 1],$$

where $g(X), g(Y)$ denote de genus of the surfaces X and Y, respectively.

Hyperelliptic Riemann surfaces

Let $h(x)$ be a polynomial of degree $2n+1+\epsilon, \epsilon \in \{0,1\}$ with distinct roots.
Then the affine plane curve X defined by $y^2 = h(x)$ is smooth.
Let $k(z) = z^{2n+2}h(1/z)$ (k is also a polynomial with distinct roots).
Form the smooth affine plane curve Y defined by the equation $w^2 = k(z)$.
Let $U = \{(x,y) \in X : x \neq 0\}$ and $V = \{(z,w) \in Y : z \neq 0\}$. (Note that U and V are open sets in X and Y, respectively). Define an isomophism

$$\begin{aligned}\phi : & U \longrightarrow V \\ & \phi(x,y) = (z,w) = (1/x, y/x^{n+1})\end{aligned}$$

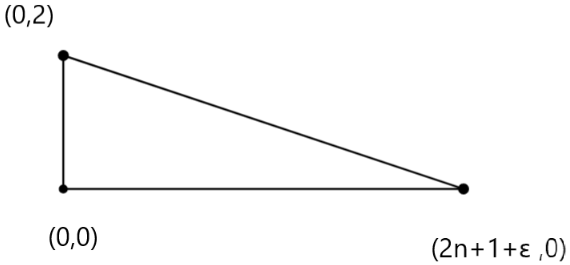
Form the compact Riemann surface Z obtained by glueing X and Y along U and V via ϕ.

$$\begin{aligned}\pi : & Z \longrightarrow \mathbb{C}_\infty \\ & (x,y) \mapsto x\end{aligned}$$

Hyperelliptic Riemann surfaces

π has degree 2 and the branch points of π are the roots of h (and the point ∞ if h has odd degree). Therefore the inverse image of any point under π is either two points with multiplicity one, or one point with multiplicity two.
By the Riemann-Hurwitz formula we have

$$2g(Z) - 2 = 2(0-2) + 2n + 2 \quad \Longrightarrow \quad g(Z) = n$$



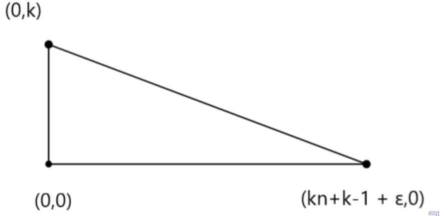
k-gonal curves

Theorem

Let $p(x)$ be a polynomial of degree $k(n+1)-1+\epsilon$ ($\epsilon \in \{0,1\}$) with distinct roots. Let X be the affine plane curve defined by $X = F^{-1}(0) \subset \mathbb{C}^2$ with

$$F(x,y) := y^k - p(x).$$

Then X admits a compactification $Z = X \cup \{\infty_1, \cdots, \infty_k\}$ with k points at infinity ($z=0$), with Z being a Riemann surface of genus $g = \frac{1}{2}(k^2(n+1) - k(n+3)) + 1$.
Moreover the genus of Z coincides with the number of points with integer coordinates in the triangle of vertices $(0,0)$, $(kn+k,0)$, $(0,k)$.



Newton Polygon

Definition

Given a Laurent polynomial

$$f(x,y) = \sum_{j=a}^{b} \sum_{k=c}^{d} a_{j.k}\, x^j y^k$$

consider the polygon defined as the convex hull of the points $(j,k) \in \mathbb{R}^2$ such that $a_{j,k} \neq 0$. To this polygon we call the Newton polygon associated to f and we denote it by $\Delta(f)$

Definition

Let $f \in \mathbb{C}[x, y, x^{-1}, y^{-1}]$ be a Laurent polynomial we define the translated polynomial $T_{k,l}[f](x,y) = x^k y^l f(x,y)$ for $k, l \in \mathbb{Z}$ and the reflected polynomial $R[f](x,y) = f(\frac{1}{x}, \frac{1}{y})$

Toric compactification

Definition

Let $f \in \mathbb{C}[x, y, x^{-1}, y^{-1}]$ be a Laurent polynomial, we define X_f as

$$X_f = f^{-1}(\{0\})$$

and $U_f = X_f \cap (\mathbb{C}^* \times \mathbb{C}^*)$.

Toric compactification

. Let $f(x, y) \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$. Write f in the following form:

$$f(x, y) = \sum_{j=a}^{b} \sum_{k=c}^{d} c_{j.k}\, x^j y^k$$

Assume $c_{a,c}, c_{b,d} \neq 0$. Define the polynomials $F(x, y) = x^{-a} y^{-c} f(x, y)$ and $G(z, w) = z^b w^d f(1/z, 1/w)$. Suppose that F and G are nonsingular. Consider the isomorphism

$$\begin{aligned} \phi : & U_F \longrightarrow U_G \\ & \phi(x, y) = (z, w) = (1/x, 1/y) \end{aligned}$$

Toric compactification

Let $X_t(f)$ be the compact Riemann surface obtained by glueing X_F and X_G along U_F and U_G via ϕ. To the type of glueing we call toric glueing and to $X_t(f)$ we call the toric compactification of the affine plane curve X_F.

Remark

When we perform this type of compactification one naturally asks how many points are we adding to X_F, that is how many ∞'s are we adding to X_F. Using the same notation as above the answer to this question is the number of distinct solutions to the equations $G(0, w) = 0$ and $G(z, 0) = 0$.

Khovanskii's Theorem

Theorem

Let $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ be a Laurent polynomial, such that we can perform the toric compactification $X_t(f)$. Then we can also perform the toric compactifications on $R[f]$ and $T_{k,l}[f]$, for any $k, l \in \mathbb{Z}$. And

$$X_t(f) \cong X_t(R[f]) \cong X_t(T_{k,l}[f])$$

Khovanskii's Theorem

Let $f \in \mathbb{C}[x, y, x^{-1}, y^{-1}]$ be a Laurent polynomial, such that we can perform the toric compactification $X_t(f)$, then we have the following equality

$$g(X_t(f)) = \# \left(int(\Delta(f)) \cap \mathbb{Z}^2 \right)$$

Where $X_t(f)$ is as above and $\Delta(f)$ is the Newton polygon of f.

Khovanskii's Theorem - Example

The rectangle

Let $k, l \in \mathbb{N}$, and $f(x,y) = \lambda + x^k + y^l + x^k y^l = 0$ with $\lambda \in \mathbb{C} \setminus \{0,1\}$.

$$y^l = -\frac{x^k + \lambda}{x^k + 1}$$

To see that f is nonsingular in X_f we verify that the system

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{cases}$$

is impossible. As before we consider new variables $z = 1/x$, $w = 1/y$ and define $g(z,w) = z^k w^l f(1/z, 1/w)$. g is also nonsingular in X_g. Now form the compact Riemann surface $X_t(f)$. Consider the holomorphic map

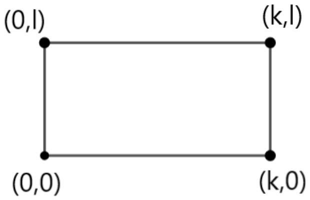
$$\pi : X_t(f) \longrightarrow \mathbb{C}_\infty$$
$$\pi(x,y) = x$$

Khovanskii's Theorem - Example

This map has degree l ($deg(\pi) = l$) and using the Lemma we see that it has $2k$ ramification points with multiplicity l. Hence by the Riemann-Hurwitz formula we have

$$g(X_t(f)) = (l-1)(k-1)$$

And the number of interior lattice points of the rectangle with vertices $(0,0), (k,0), (0,l), (k,l)$



Isomorphic Riemann Surfaces and affine transformations

Definition

A $\mathbb{Z}$-affine transformation is a map

$$\begin{aligned}\psi : & \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ & x \mapsto Ax + b\end{aligned}$$

Where $A \in GL_2(\mathbb{Z})$ and $b \in \mathbb{Z}^2$.

Definition

Let $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, (i,j) \mapsto (k(i,j), l(i,j))$ be a $\mathbb{Z}$-affine transformation and $f(x,y) = \sum_{j=a}^{b} \sum_{j=c}^{d} c_{i,j} x^i y^j \in \mathbb{C}[x, y, x^{-1}, y^{-1}]$ a Laurent polynomial we define $\psi[f]$ as

$$\psi[f](x,y) = \sum_{j=-a}^{b} \sum_{j=-c}^{d} c_{i,j} x^{k(i,j)} y^{l(i,j)}$$

Isomorphic Riemann Surfaces and affine transformations

Theorem

Let $f \in \mathbb{C}[x, y, x^{-1}, y^{-1}]$ be a nonsingular Laurent polynomial such that and ψ be an $\mathbb{Z}$-affine transformation and let $\psi[f]$ be as before, then

$$X_{\psi[f]} \cong X_f$$

Proof (idea)

We need only to verify the result for translations and the $\mathbb{Z}$-affine transformations associated to the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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