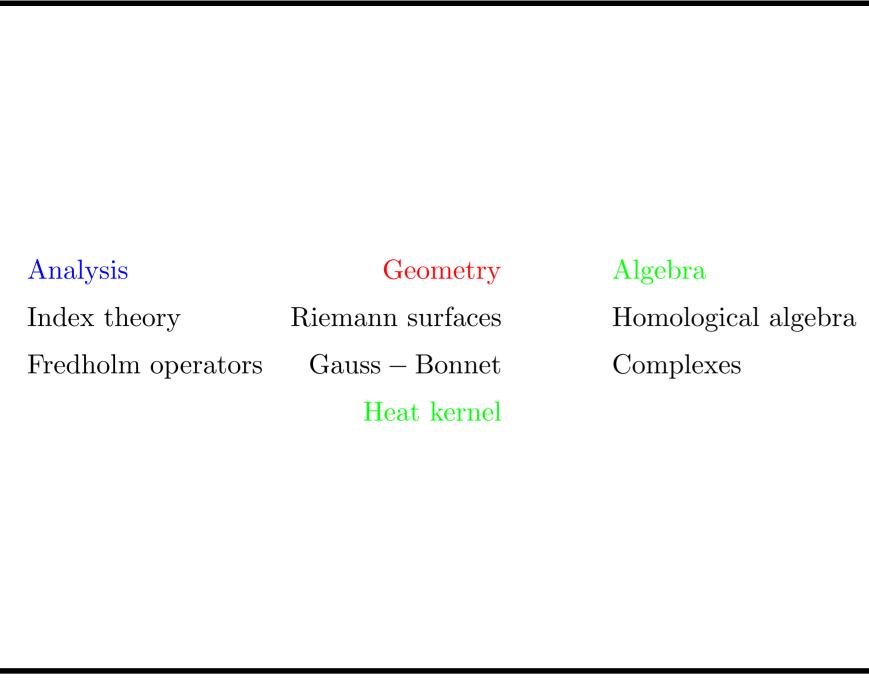
Some connexions between analysis and geometry Jean-Michel Bismut Université Paris-Sud Orsay

Luso, September 20th 2003



Fredholm operators

•E, F Banach spaces.

• $A \in \mathcal{L}(E, F)$ Fredholm if A has finite dimensional kernel and cokernel.

• $A \in \mathcal{L}(E, F)$ Fredholm if and only if A is invertible modulo compact operators:

$$TA = 1 + K, \qquad AT = 1 + K'.$$

• $\mathcal{F}(E,F)$ is open in $\mathcal{L}(E,F)$.

•ind $A = \dim \ker A - \dim \operatorname{coker} A \in \mathbb{Z}$ (index of A) is a homotopy invariant.

Examples

1) If E, F finite dimensional, ind $(A) = \dim E - \dim F$. 2) If $E = F = L_2^{\geq 0}$ ([0, 1]) (Fourier series with Fourier coefficients $a_n, n \geq 0$), f(z) smooth $S_1 \to \mathbb{C}^*$, with $S_1 \simeq \mathbb{R}/\mathbb{Z}$.

 $A_f = P^{\ge 0} f.$

Then A_f is Fredholm, and

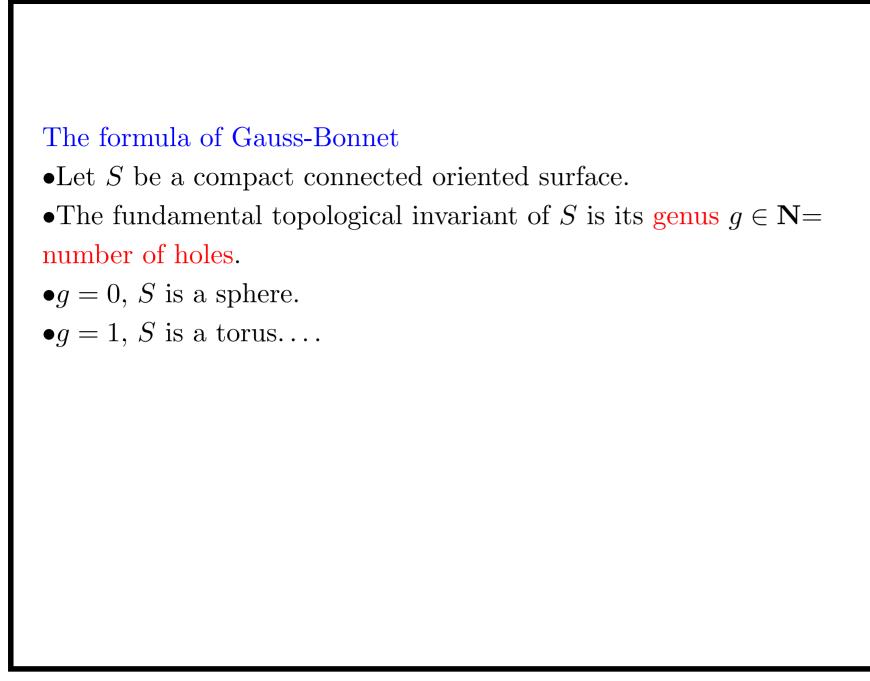
ind $(A_f) = -$ rotation number of f.

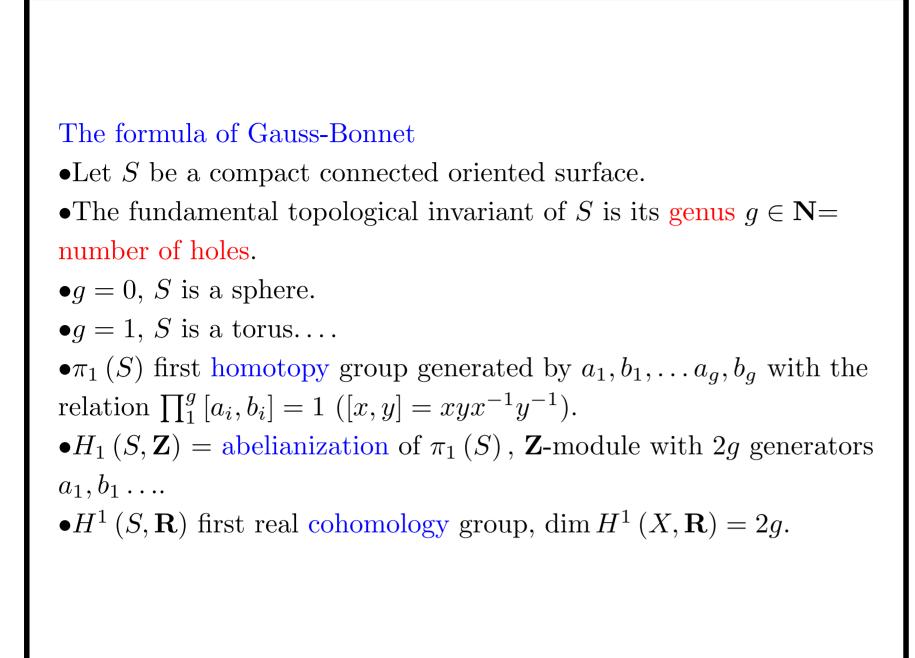
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The index has been expressed in topological terms. (work out the example $f(z) = z^k \dots$) Question (Gelfand): given an elliptic pseudodifferential operator Aon M compact manifold, find a formula for ind A. Answer: Atiyah-Singer index theorem.





•Euler characteristic $\chi(S) = \sum_{i=1}^{2} (-1)^{i} \dim H^{i}(S, \mathbf{R}) = 2 - 2g.$ Geometric interpretation of $\chi(S)$ • $\chi(S) = S \cap S$ in $S \times S$ (self-intersection of the diagonal). •Y vector field on S with isolated nondegenerate zeroes. Poincaré-Hopf: $\chi(S) = \sum_{\text{zeroes of } Y} (-1)^{\text{ind}(x)}.$ (follows from the above).

•If S embedded in \mathbb{R}^3 , the degree of the Gauss map $x \in S \to n(x) \in S_2$ is exactly 1 - g (direct computation, or use an affine Morse function f and its gradient field X).

•If S embedded in \mathbb{R}^3 , the degree of the Gauss map $x \in S \to n(x) \in S_2$ is exactly 1 - g (direct computation, or use an affine Morse function f and its gradient field X). •Gauss-Bonnet: If S embedded surface in \mathbb{R}^3 , K scalar curvature (= 2 for the sphere S_2 !),

$$\chi\left(S\right) = \int_{S} \frac{K}{4\pi} dx.$$

Use the fact that $\int_{S} \frac{K}{2} dx = \deg(n) \operatorname{Vol}(S_2)$. •The previous equality remains valid for any Riemannian metric on S (Gauss).

Gauss-Bonnet and Poincaré-Hopf

• Y a generic section of TS.

•Set

$$\alpha_t = \frac{1}{2\pi} \exp\left(-t \left|Y\right|^2 / 2\right) \left(\frac{K}{2} dx + t\omega \left(\nabla^{TS} Y, \nabla^{TS} Y\right)\right)$$

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•The α_t are closed (!) cohomologous 2-forms [3] (Mathai-Quillen). • $\alpha_0 = \frac{K}{4\pi} dx, \alpha_{+\infty} = \sum_{\text{zeroes of } Y} (-1)^{\text{ind}(x)} \delta_x.$ •We have given another proof of Poincaré-Hopf.

The Koszul complex

- E is a two dimensional real vector space.
- • E^* dual of E, generic element $\xi \in E^*$.
- •Creation $\xi \wedge$, annihilation i_X act on $\Lambda^{\cdot}(E^*)$,

 $[\xi \wedge, i_X] = \langle \xi, X \rangle \,.$

•Koszul complex

$$0 \to \Lambda^0 E^* = \mathbf{R} \xrightarrow{\xi \wedge} \Lambda^1 E^* \xrightarrow{\xi \wedge} \Lambda^2 E^* \to 0$$

is acyclic for $\xi \neq 0$ (take $X \in E, \langle \xi, X \rangle = 1$).

•If E equipped with scalar product, one can take $X = \xi^*$, and use $[\xi \wedge, i_{\xi^*}] = |\xi|^2$.

The de Rham complex

 $\bullet\left(\Omega^{\cdot}\left(S\right),d\right)$ de Rham complex

 $0 \to \Omega^0\left(S\right) \xrightarrow{d} \Omega^1\left(S\right) \xrightarrow{d} \Omega^2\left(S\right) \to 0.$

 $\begin{aligned} \bullet d &= dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y}, \sigma \left(d \right) = i\xi \land \left(\sigma \left(d \right) = \text{principal symbol} \right). \\ \bullet d^2 &= 0, \left(\Omega^{\cdot} \left(S \right), d \right) \text{ elliptic complex, cohomology } \left(\ker d / \text{Im } d \right) \\ &\simeq H^{\cdot} \left(S, \mathbf{R} \right). \end{aligned}$

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•Hodge theory. g^{TS} a metric on TS. Scalar product on $\Omega^{\cdot}(S)$,

$$\langle s, s' \rangle = \int_M \langle s, s' \rangle \, dx.$$

• d^* adjoint of d.

 $\Box = dd^* + d^*d = (d + d^*)^2$

is the Laplacian: elliptic operator of order 2. <u>Theorem</u> (Hodge) ker $\Box = \ker (d + d^*) \simeq H^{\cdot}(S, \mathbf{R}).$

- • $d + d^*$ exchanges $\Omega^{\text{even}}(S)$ and $\Omega^{\text{odd}}(S)$.
- $(d + d^*)|_{\text{even}}$ is Fredholm, and $\operatorname{ind} (d + d^*)|_{\text{even}} = \chi(S)$ (elementary...)
- •For any t > 0, by McKean-Singer [4],

$$\chi(S) = \operatorname{Tr}_{s}\left[\exp\left(-t\Box\right)\right] = \int_{S} \operatorname{Tr}_{s}\left[P_{t}\left(x,x\right)\right] dx.$$

 $(Tr_s \text{ supertrace, use spectral theory...})$

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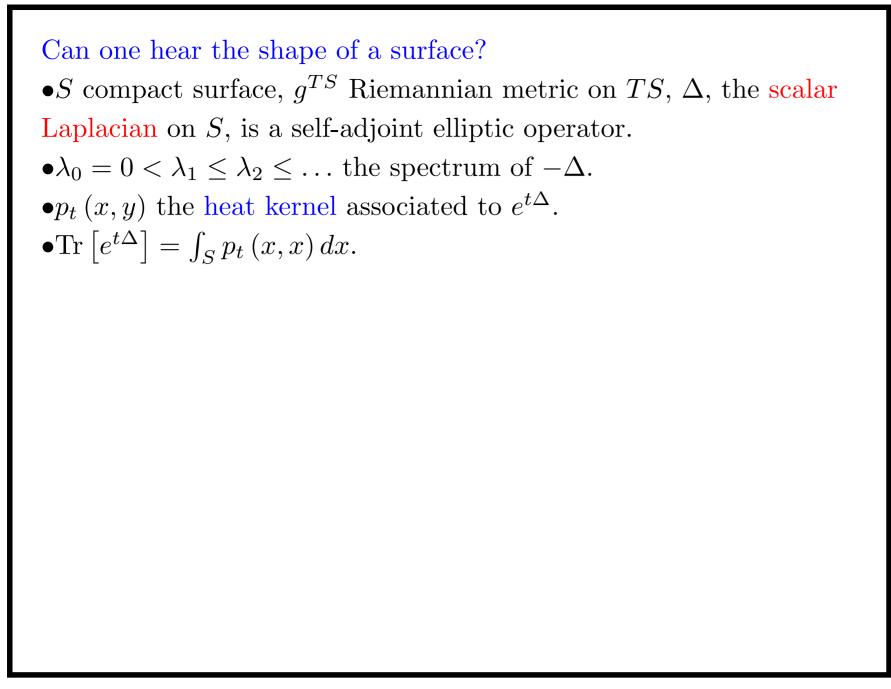
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•Weitzenböck formula:

$$\Box = -\Delta + \frac{K}{4}N - \frac{K}{2}Q,$$

- N number operator, Q = 1 on 2 forms.
- •As $t \to 0$, $\operatorname{Tr}_{s} \left[P_{t}(x,x) \right] \simeq \frac{1}{4\pi t} \operatorname{Tr}_{s} \left[e^{-tK(x)N/4 + K(x)Q/2} \right]$. • $\operatorname{Tr}_{s} \left[e^{-tK(x)N/4 + tK(x)Q/2} \right] = 1 - 2e^{-tK/4} + e^{tK/2} \simeq tK$.
- •We get $\chi(S) = \int_{S} \frac{K}{4\pi} dx$ (Gauss-Bonnet, cancellations in local index theory [1]).



Can one hear the shape of a surface? • S compact surface, g^{TS} Riemannian metric on TS, Δ , the scalar Laplacian on S, is a self-adjoint elliptic operator. • $\lambda_0 = 0 < \lambda_1 \le \lambda_2 \le \ldots$ the spectrum of $-\Delta$. • $p_t(x, y)$ the heat kernel associated to $e^{t\Delta}$. • $\operatorname{Tr}\left[e^{t\Delta}\right] = \int_S p_t(x, x) \, dx$. • As $t \to 0$, $p_t(x, x) = \frac{1}{4\pi t} \left(1 + \frac{tK(x)}{6} + \mathcal{O}_x(t^2)\right)$. • As $t \to 0$, $\operatorname{Tr}\left[e^{t\Delta}\right] = \frac{1}{4\pi t} \left(\operatorname{Vol}(S) + \frac{t}{6} \int_S K dx\right)$, equivalent to $\operatorname{Tr}\left[e^{t\Delta}\right] = \frac{\operatorname{Vol}(S)}{4\pi t} + \frac{\chi(S)}{6} + \dots$

Conclusion

•One can hear the volume of S.

•One can hear the genus g of S.

•Special role played by the constant terms in the asymptotic expansion.

- •Note that $B_1 = \frac{1}{6}$ is the first Bernouilli number. •Todd series $\operatorname{Td}(x) = \frac{x}{1-e^{-x}} = 1 + \frac{x}{2} + \frac{B_1}{2}x^2 + \dots$

Heat kernel and the loop space •Heat semigroup $e^{(t+t')\Delta} = e^{t\Delta}e^{t'\Delta}$. • $e^{t\Delta} = e^{t\Delta/n} \dots e^{t\Delta/n}$.

•Another expression for the trace

$$\operatorname{Tr}\left[e^{t\Delta}\right] = \int_{X^n} \underbrace{p_{t/n}\left(x_0, x_1\right) \dots p_{t/n}\left(x_{n-1}, x_0\right)}_{A} dx_0 \dots dx_{n-1}.$$

cyclic expression for the trace.

Heat kernel and the loop space

•Heat semigroup $e^{(t+t')\Delta} = e^{t\Delta}e^{t'}\Delta$. • $e^{t\Delta} = e^{t\Delta\nabla} \dots e^{t\Delta/n}$.

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cyclic expression for the trace.

•Compare to

$$\operatorname{Tr}[A^n] = \sum a_{i_0 i_1} a_{i_1 i_2} \dots a_{i_{n-1} i_0}$$

•The above sum is a sum on discrete closed loops.

•As $n \to +\infty$, the integral 'converges' to an integral on... the loop space of X.

•This measure is the Wiener measure on LX, it is invariant par rotations.

The Migdal invariant of a surface •*S* a compact oriented surface of genus *g*. •*K* a triangulation of *S*, with *a* edges. •*A*_{σ} > 0 the area of the simplex σ . •SU (2) \simeq *S*₃ the group of special unitary transformations of \mathbb{C}^2 , *p*_t (*g*) the heat kernel on *G*.

The Migdal invariant of a surface

• S a compact oriented surface of genus g.

• K a triangulation of S, with a edges.

• $A_{\sigma} > 0$ the area of the simplex σ .

•SU(2) $\simeq S_3$ the group of special unitary transformations of \mathbb{C}^2 , $p_t(g)$ the heat kernel on G.

•To each oriented edge of K, we associate an element $g \in SU(2)$.

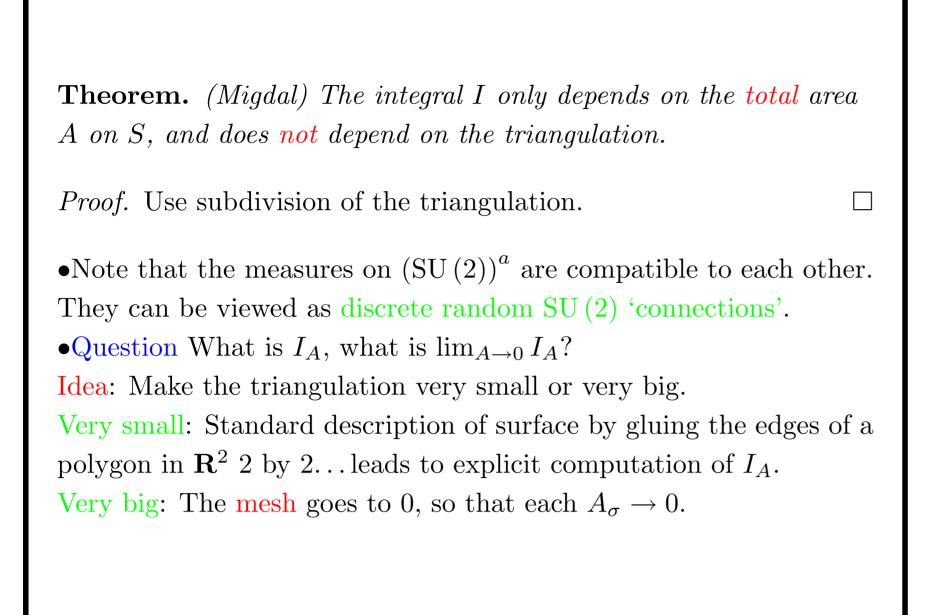
•Each simplex σ has a holonomy $H_{\sigma} \in SU(2)$ (ordered product of the group elements of the edges), well defined up to conjugation.

• $p_{A_{\sigma}}(H_{\sigma}) > 0$ is well-defined.

 $\bullet Set$

$$I = \int_{G^a} \prod p_{A_{\sigma}} (H_{\sigma}) \, dg_1 \dots dg_a.$$

Theorem. (Migdal) The integral I only depends on the total area A on S, and does not depend on the triangulation.
Proof. Use subdivision of the triangulation.
□



•For $B \in su(2)$, |B| small, as $t \to 0$,

$$p_t\left(e^B\right) \simeq \frac{\exp\left(-\left|B\right|^2/4t\right)}{\left(4\pi t\right)^{3/2}}.$$

•If each simplex has area t/n,

$$I_t \simeq C_t \int \exp\left(-n\sum |B_\sigma|^2/4t\right).$$

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•

•If A is a connection and F^A its curvature, the holonomy of a path bounding a domain D of small area a is $\simeq \exp(aF^A)$. •If $B_{\sigma} \simeq \frac{t}{n}F^A$, then

$$n\sum |B_{\sigma}|^2 / 4t \simeq \frac{\int_S |F^A|^2}{4t}.$$

... Yang-Mills functional.

•We find that

$$I_t = \int_{\mathcal{A}} \exp\left(-\frac{\int_S |F^A|^2}{4t}\right) d\mathcal{A},$$
=partition function for the Yang-Mills model.
•As $t \to 0$, the integral localizes on the space of flat connections.

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•As $t \to 0$, the integral localizes on the space of flat connections. **Theorem.** (Witten [5])

 $\lim_{t\to 0} I_t = \text{symplectic volume of } \mod \text{moduli space of flat connections.}$

A proof of Witten result [2]

 $\bullet f: G^{2g} \to G$ the map

$$(u_1, v_1, \ldots, u_g, v_g) \rightarrow \prod_{i=1}^g [u_i, v_i].$$

• G acts on G^{2g} and G by conjugation.

• f is a G equivariant map

$$f\left(g.x\right) = g.f\left(x\right).$$

Question: What is the image of the measure $dg_1 \dots dg_{2g}$ by f? Answer: Change of variable formula.

(1)
$$f(g.x) = g.f(x).$$

•Differentiate (1) in the variable g at g = 1, when f(x) = 1. If $A \in su(2)$,

(2)
$$\langle df(x), A.x \rangle = 0.$$

 $\bullet \mathrm{We}$ have the finite dimensional complex,

(3)
$$0 \to su(2) \xrightarrow{\partial} su(2)^{2g} \xrightarrow{\partial} su(2) \to 0.$$

•The Euler characteristic of this complex is 3(2-2g)...

Explanation

•The set $\{x \in G^{2g}, f(x) = 1\}$ is the set of representations of $\pi_1(S) \to G$.

•The complex

$$0 \to su\left(2\right) \stackrel{\partial}{\longrightarrow} su\left(2\right)^{2g} \stackrel{\partial}{\longrightarrow} su\left(2\right) \to 0$$

is a combinatorial complex whose cohomology is the cohomology of the flat adjoint bundle on S.

•One can now use the standard change of variables formula....

References

- [1] M. Atiyah, R. Bott, and V. K. Patodi. On the heat equation and the index theorem. *Invent. Math.*, 19:279–330, 1973.
- [2] J.-M. Bismut and F. Labourie. Symplectic geometry and the Verlinde formulas. In Surveys in differential geometry: differential geometry inspired by string theory, pages 97–311.
 Int. Press, Boston, MA, 1999.
- [3] V. Mathai and D. Quillen. Superconnections, Thom classes, and equivariant differential forms. *Topology*, 25(1):85–110, 1986.
- [4] H. P. McKean, Jr. and I. M. Singer. Curvature and the eigenvalues of the Laplacian. J. Differential Geometry, 1(1):43-69, 1967.
- [5] E. Witten. On quantum gauge theories in two dimensions. Comm. Math. Phys., 141:153–209, 1991.