Notes by Richard Taylor

10 Function Spaces

Many of the ideas of linear algebra, which we have studied in the context of $\mathbb{R}^n$ or $\mathbb{C}^n$, are applicable much more widely in the mathematical sciences. To try to capture the domain of validity of these methods, mathematicians introduce the concept of "vector space" or "linear space". (These two terms are synonyms.) Rather than studying linear spaces in the abstract, we shall look at some examples which are important in the theory of differential equations.

10.1 Ordinary Linear Differential Equations

(Compare this section with sections 4.1, 4.2 and 9.3 in the book)

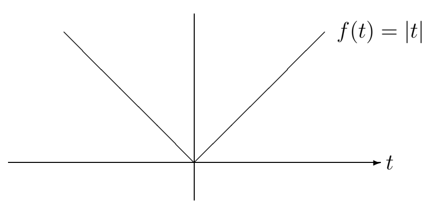
By a smooth function from the real numbers to themselves we shall mean a function $f : \mathbb{R} \to \mathbb{R}$ which can be differentiated as many times as you like. We will denote the set of all such functions by C^∞. For instance

$$\begin{aligned} f(t) &= 1 \\ g(t) &= t \\ h(t) &= e^t \end{aligned}$$

are all smooth functions. Indeed

$$\frac{d^n f}{dt^n} = 0 \text{ for all } n > 0$$
$$\frac{dg}{dt} = 1 \; ; \; \frac{d^n g}{dt^n} = 0 \text{ for all } n > 1$$
$$\frac{d^n h}{dt^n} = e^t \text{ for all } n > 0.$$

On the other hand $f(t) = |t|$ is not a smooth function as it is not even once differentiable at $t = 0$.



If $c \in \mathbb{R}$ and if f and g are smooth functions, so is $(cf+g)(t) = cf(t)+g(t)$. (Recall that if f and g are differentiable, so is $cf + g$ and $(cf + g)'(t) = cf'(t) + g'(t)$.)

Thus for example $1 + t$ and $t + 2e^t$ are in our collection C^∞.

Thus on our collection of functions C^∞, we have defined two operations:

(a) "addition" e.g. if $f(t) = 1$ and $h(t) = e^t$ then $(f + h)(t) = 1 + e^t$;

(b) and "scalar multiplication" e.g. if $h(t) = e^t$ then $(2h)(t) = 2e^t$.

These are the same basic operations that we have studied on $\mathbb{R}^n$. Just as most of our study of $\mathbb{R}^n$ was immediately applicable to $\mathbb{C}^n$, so many of the same ideas also apply to our new "space" $\mathbb{C}^n$. More specifically $\mathbb{C}^n$ is a "linear space" in the sense of section 9.1.

Examples

(1) Any polynomial function $a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ is smooth and so the collection of all polynomial functions forms a subset P of C^∞. This subset P has the following two important properties

   (a) If $f(t)$, $g(t)$ are polynomials so is $f(t) + g(t)$

   (b) If $f(t)$ is a polynomial and c is a real number then $cf(t)$ is a polynomial.

   Because P has properties (a) and (b) we call P a subspace of C^∞.

(2) Suppose c_1 and c_2 are real numbers and $c_1 t + c_2 e^t$ is the zero function. Then we must have that $c_1 = c_2 = 0$. (Why? If $c_2 \neq 0$ then for t very large and positive $c_2 e^t$ will be much larger than $c_1 t$ in magnitude and so $c_1 t + c_2 e^t \neq 0$. Thus one must have $c_2 = 0$ and hence also $c_1 = 0$.) Because of this property we say that t and e^t are linearly independent.

(3) On the other hand
$$1.1 + 1.t + (-1).(1 + t) = 0$$
and so we say that
$$1, t \text{ and } (1 + t)$$
are linearly dependant.

(4) If $f(t) \in C^\infty$ we define a new function $(Df)(t)$ by
$$(Df)(t) = f'(t) = \frac{df}{dt}(t).$$
(For example $D(\sin(t)) = \cos(t)$.) In general $D(f)$ is again a smooth function so D gives a function from C^∞ to C^∞. (D is a "function of functions".) Moreover D has the following two important properties:

   (a) $D(f(t) + g(t)) = D(f(t)) + D(g(t))$

   (b) $D(cf(t)) = cD(f(t))$,

   whenever $c \in \mathbb{R}$; $f(t), g(t) \in C^\infty$. Because D has these two properties, we call D a linear transformation or we simply say D is linear.

(5) What is the kernel of D? It is simply the collection of functions $f(t) \in C^\infty$ such that $D(f(t)) = 0$. But the only functions with zero derivative are the constant functions. Thus $\ker(D)$ is the collection all constant functions.

(6) What is the image of D? It is the whole of C^∞. Why? If $f(t) \in C^\infty$ then we can define a new function

$$g(t) = \int_0^t f(s)ds.$$

Then $g(t)$ is also a smooth function and by the fundamental theorem of calculus $D(g(t)) = f(t)$.

(7) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and $\text{Im } T = \mathbb{R}^n$ then $\ker T = (0)$. However $D : C^\infty \to C^\infty$ is a linear transformation and $\text{Im } D = C^\infty$, but $\ker D \neq (0)$. This can happen because C^∞ is "infinite dimensional", by which we mean that C^∞ cannot be spanned by any finite number of elements $f_i(t) \in C^\infty$.

(8) Now consider $D^2 = D \circ D : C^\infty \to C^\infty$. Its kernel is the collection of smooth functions $f(t)$ such that $f''(t) = 0$, i.e. such that $f'(t) = a$, a constant, i.e. such that

$$f(t) = at + b$$

for some real numbers a, b. Thus any element $f(t) \in \ker D^2$ is a linear combination of t and 1, i.e.

$$f(t) = a.t + b.1$$

We say that t and 1 span $\ker D^2$.

(9) In fact the functions t and 1 are also linearly independant and so we say that they form a basis of $\ker D^2$. As this basis has two elements we say that $\ker D^2$ is two dimensional.

(10) Find all solutions $f(t)$ of the equation $D^2 f(t) = e^t$.

It is not too hard to spot that $f_0(t) = e^t$ is one solution of this equation. If $f(t)$ is any other solution then $D^2\big(f(t) - f_0(t)\big) = D^2(f(t)) - D^2(f_0(t)) = e^t - e^t = 0$. On the other hand if $D^2\big(f(t) - f_0(t)\big) = 0$ then $D^2(f(t)) = D^2(f_0(t)) = e^t$.
Thus $f(t)$ is a solution of $D^2 f(t) = e^t$ if and only if $f(t) - f_0(t) \in \ker D^2$. Thus the general solution is

$$f(t) = f_0(t) + at + b = e^t + at + b.$$

Just as for linear equations, to find the general solution of an inhomogeneous equation (eg. $D^2 f(t) = e^t$) you find any solution and add to it a general solution of the corresponding homogeneous equation (eg. $D^2 f(t) = 0$).

Let us take this opportunity to explain what we mean by saying a sequence $f_1(t), f_2(t), \ldots$ of elements of a subspace $V \in C^\infty$ span V. We will mean that if $g(t)$ is any element of V then we can find a finite number of real numbers $c_1, \ldots, c_n$ such that

$$g(t) = c_1 f_1(t) + \cdots + c_n f_n(t).$$

The sequence $f_1(t), f_2(t), \ldots$ may be infinite, but we require that any $g(t)$ is a linear combination of only finitely many $f_1(t), \ldots, f_n(t)$. The number n we need may depend on $g(t)$. For example $1, t, t^2, t^3, \ldots$ spans P but does not span C^∞. (See exercises for 9.1.)

Suppose that $a_n(t), \ldots, a_0(t), g(t) \in C^\infty$. Then we will refer to an equation of the form

$$a_n(t)\frac{d^n f(t)}{dt^n} + a_{n-1}(t)\frac{d^{n-1} f(t)}{dt^{n-1}} + \cdots + a_0(t) f(t) = g(t) \qquad (*)$$

as a linear ordinary differential equation. To the equation $(*)$ we may associate a linear transformation

$$T : C^\infty \to C^\infty$$

defined by

$$T\big(f(t)\big) = a_n(t)\frac{d^n f(t)}{dt^n} + \cdots + a_1(t)\frac{df}{dt} + a_0(t) f(t).$$

It is easy to check that T is indeed a linear transformation. The equation $(*)$ can be rewritten

$$T\big(f(t)\big) = g(t).$$

If $g(t) \neq 0$ we will call this equation inhomogeneous. If $g(t) = 0$ we will call it homogeneous. We will refer to the associated equation

$$T\big(f(t)\big) = 0$$

as the associated homogeneous equation.

If $f_0(t)$ is any given solution of

$$T\big(f(t)\big) = g(t)$$

then the general solution is

$$f(t) = f_0(t) + h(t)$$

where $h(t) \in \ker T$, i.e. $h(t)$ is a solution of the associated homogeneous equation.

We have the following two important facts which guarantee the existence of solutions of certain linear ODE's (Ordinary Differential Equations)

Fact 10.1.1. *Suppose*

$$T\big(f(t)\big) = \frac{d^n f(t)}{dt^n} + a_{n-1}(t)\frac{d^{n-1} f(t)}{dt^{n-1}} + \cdots + a_1(t)\frac{df(t)}{dt} + a_0(t) f(t)$$

is a linear transformation from C^∞ to C^∞. Then $\ker T$ has dimension n.

Fact 10.1.2. *Suppose*

$$T\big(f(t)\big) = \frac{d^n f(t)}{dt^n} + a_{n-1}(t)\frac{d^{n-1} f(t)}{dt^{n-1}} + \cdots + a_1(t)\frac{df(t)}{dt} + a_0(t) f(t)$$

and suppose $g(t) \in C^\infty$. Then there exists $f(t) \in C^\infty$ with

$$T\big(f(t)\big) = g(t).$$

Note that in both these facts we are assuming that the coefficient of $\frac{d^n f(t)}{dt^n}$ are 1. Both facts become false if we do not assume this. For instance if $T\big(f(t)\big) = tf'(t) + f(t)$ then $\dim(\ker T) = 0$. Also if $S\big(f(t)\big) = tf'(t)$ then there is no function $f(t) \in C^\infty$ with $S\big(f(t)\big) = 1$. (See the exercises.)

For example consider the case of "constant coefficients", i.e.

$$T = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + A_0$$

where $a_{n+1}, \ldots, a_0 \in \mathbb{R}$. It is convenient to look at the polynomial

$$X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0,$$

sometimes, if slightly confusingly, called the characteristic polynomial of T. Over the complex numbers we may factorise this polynomial

$$(X - \alpha_1)(X - \alpha_2)\ldots(X - \alpha_n)$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}^n$; and we may also factorise

$$T = (D - \alpha_1)\ldots(D - \alpha_n).$$

If $\alpha_j \in \mathbb{R}$ then $\frac{d}{dt}e^{\alpha_j t} = \alpha_j e^{\alpha_j t}$ so $(D - \alpha_j)e^{\alpha_j t} = 0$, so that $Te^{\alpha_j t} = 0$, i.e. $e^{\alpha_j t} \in \ker T$.

For instance if $T = D^2 - D - 2$, then the "characteristic polynomial" is $X^2 - X - 2 = (X-2)(X+1)$. Thus $T = (D-2)(D+1)$ and so e^{2t} and e^{-t} are in $\ker T$. As $\ker T$ has dimension 2 by Fact 10.1.1 we see that e^{2t} and e^{-t} form a basis of $\ker T$, i.e. $\ker T$ is the collection of all functions $c_1 e^{2t} + c_2 e^{-t}$.

If on the other hand $\alpha_j \in \mathbb{C}$ but is not real then the complex conjugate $\overline{\alpha_j}$ of α_j is also a root of $X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$. Write

$$\begin{aligned} \alpha_j &= a + ib \\ \overline{\alpha_j} &= a - ib. \end{aligned}$$

Then $Te^{\alpha_j t} = 0$, but now $e^{\alpha_j t}$ is not in C^∞ as it is not real valued.

$$e^{\alpha_j t} = e^{at}(\cos bt + i \sin bt).$$

Also $Te^{\overline{\alpha_j}t} = 0$ and

$$e^{\overline{\alpha_j}t} = e^{at}(\cos bt - i \sin bt).$$

Thus

$$T\left(\frac{1}{2}(e^{\alpha_j t} + e^{\overline{\alpha_j}t})\right) = 0 \text{ i.e. } T(e^{at}\cos bt) = 0$$

and

$$T\left(\frac{1}{2i}(e^{\alpha_j t} - e^{\overline{\alpha_j}t})\right) = 0 \text{ i.e. } T(e^{at}\sin bt) = 0;$$

i.e. $e^{at}\cos bt$ and $e^{at}\sin bt \in \ker T$.

For instance if $T = D^2 - 6D + 13$ then the "characteristic polynomial" is $X^2 - 6X + 13$ which has roots

$$3 \pm \sqrt{-4} = 3 \pm 2i.$$

Thus $e^{3t}\cos 2t$ and $e^{3t}\sin 2t \in \ker T$. As $\dim\ker T = 2$ we see that $e^{3t}\cos 2t$ and $e^{3t}\sin 2t$ is a basis of $\ker T$, i.e. the general solution of

$$T\big(f(t)\big) = 0$$

is

$$c_1 e^{3t}\cos 2t + c_2 e^{3t}\sin 2t.$$

Suppose now we are asked to find the general solution of

$$T\big(f(t)\big) = 30\cos t.$$

We must first look for some particular solution to this equation. Experience can teach us that a good bet is to look for a solution

$$f(t) = A\cos t + B\sin t.$$

Then

$$\begin{aligned} T\big(f(t)\big) &= -A\cos t - B\sin t + 6A\sin t - 6B\cos t + 13A\cos T + 13B\sin t \\ &= (12A - 6B)\cos t + (6A + 12B)\sin t. \end{aligned}$$

This will give a solution to

$$T\big(f(t)\big) = 30\cos t$$

if and only if

$$\begin{aligned} 12A - 6B &= 30 \\ 6A + 12B &= 0 \end{aligned}$$

i.e.

$$A = 2 \qquad B = -1.$$

Thus we have found a particular solution

$$f(t) = 2\cos t - \sin t.$$

We deduce that the general solution is

$$f(t) = 2\cos t - \sin t + c_1 e^{3t}\cos 2t + c_2 e^{3t}\sin 2t$$

WE RECOMMEND YOU ALSO READ SECTION 9.3

EXERCISES

(1) which of the following sets are subspaces of C^∞? Justify your answers.

(a) All continuous functions from $\mathbb{R}$ to $\mathbb{R}$.
(b) All $f \in C^\infty$ such that $f(0) + f'(0) = 0$.
(c) All $f \in C^\infty$ such that $f + f' = 0$.
(d) All $f \in C^\infty$ such that $f(0) = 1$.

(2) Which of the following subsets of C^∞ are LI? Justify your answers.

(a) $1, t, t^2, t^3$.
(b) $1 + t, 1 - t, t^2, 1 + t + t^2$.
(c) $\sin t, e^t, e^{-t}$.
(d) $\sin t, \cos t, \sin(t + \pi/3)$.

(3) Which of the following functions are linear? Justify your answers.

(a) $T : C^\infty \to \mathbb{R}; \quad T(f) = f(0)$.
(b) $T : C^\infty \to C^\infty; \quad T(f) = f^2 + f'$.
(c) $T : C^\infty \to \mathbb{R}^2; \quad T(f) = \begin{bmatrix} f(0) \\ f(1) \end{bmatrix}$.
(d) $T : C^\infty \to \mathbb{R}; \quad T(f) = \int_0^1 f(t)dt$.

(4) Find a basis for the kernel of $T : C^\infty \to C^\infty$ given by

$$T(f)(t) = f''(t) - f(0).$$

(5) Find a basis for the image of $T : C^\infty \to C^\infty$ given by

$$T(f)(t) = f(0) + f'(0)t + (f(0) + f'(0))t^2.$$

(6) Find the eigenvalues and eigenspaces for $T : C^\infty \to C^\infty$ given by $T(f) = f' + f$.

(7) Let $T\big(f(t)\big) = f''(t) + f'(t) - 12f(t)$. Find a basis for $\ker T$. Find a smooth function $f(t)$ such that

$$\begin{aligned} &T\big(f(t)\big) = 0 \\ \text{and} \quad &f(0) = f'(0) = 0. \end{aligned}$$

(8) Let $T\big(f(t)\big) = f''(t) + 2f'(t) + 2f(t)$. Find a basis for $\ker T$. Find a smooth function $f(t)$ such that

$$\begin{aligned} &T\big(f(t)\big) = 0 \\ \text{and} \quad &f(0) = f'(0) = 1. \end{aligned}$$

(9) Problem 34 of section 9.3

(10) Let $T\big(f(t)\big) = f''(t) + 9f(t)$. Find a basis for $\ker T$. Also find the general solution of

$$T\big(f(t)\big) = \cos(\alpha t)$$

where α is a positive real number. Distinguish the cases $\alpha = 3$ and $\alpha \neq 3$. [HINT: In the case $\alpha = 3$ consider $At \cos 3t + Bt \sin 3t$.]

(11) Solve the equation $t\dfrac{df(t)}{dt} = 1$ and explain why it has no solution in C^∞.

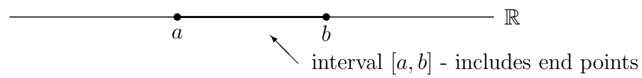
(12) Let $T\big(f(t)\big) = tf'(t) + f(t)$. Suppose $T\big(f(t)\big) = 0$. If $g(t) = tf(t)$ show that $g'(t) = 0$. Conclude that $\dim(\ker T) = 0$.

10.2 Fourier Series

(Compare this with section 5.5.) In the last section we looked at spaces of functions which behaved like $\mathbb{R}^n$, but we did not look at any analogues of the concepts of length, angle or dot product. In this section we will discuss an example in which the analogues of these concepts play an important role.

Recall that if a and b are real numbers with $a < b$ then $[a, b]$ denotes the interval

$$\{x \in \mathbb{R} : a \leq x \leq b\}.$$

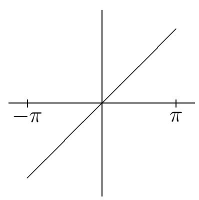


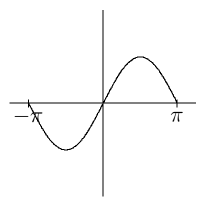
We will let $C[-\pi, \pi]$ denote the collection of all continuous functions from the interval $[-\pi, \pi]$ to $\mathbb{R}$.

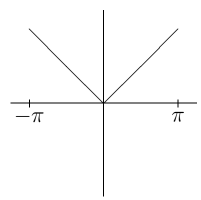
For example

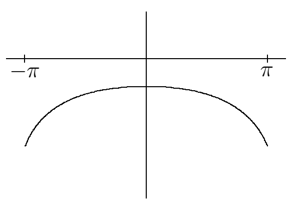
$$t, \sin t, |t|, \frac{1}{t^2 - 16}$$

are all functions in $C[-\pi, \pi]$





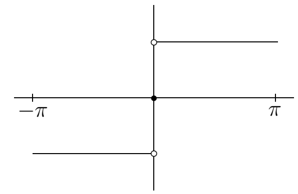




On the other hand the function

$$f(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

does not lie in $C[-\pi, \pi]$



Again $C[-\pi, \pi]$ is a linear space:

(a) If $f(t)$ and $g(t) \in C[-\pi, \pi]$ then $f(t) + g(t) \in C[-\pi, \pi]$ (recall that the sum of continuous functions is continuous), eg. $|t| + \sin t \in C[-\pi, \pi]$

(b) If $f(t) \in C[-\pi, \pi]$ and $c \in \mathbb{R}$ then $cf(t) \in C[-\pi, \pi]$, eg. $2 \sin t \in C[-\pi, \pi]$.

We will define the inner product of two functions $f(t), g(t) \in C[-\pi, \pi]$ to be

$$\langle f(t), g(t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)g(t)dt.$$

You should think of it as an analogue of the dot product of two vectors in $\mathbb{R}^n$.
It shares with the dot product the following three key properties:

(1) If $f(t)$ and $g(t) \in C[-\pi, \pi]$ then

$$\langle f(t), g(t) \rangle = \langle g(t), f(t) \rangle$$

(2) If $f(t)$, $g(t)$ and $h(t) \in C[-\pi, \pi]$ and if $c \in \mathbb{R}$ then

$$\langle cf(t) + g(t), h(t) \rangle = c\langle f(t), h(t) \rangle + \langle g(t), h(t) \rangle$$

(3) If $f(t) \in C[-\pi, \pi]$ is non-zero (i.e. not identically zero) then

$$\langle f(t), f(t) \rangle > 0.$$

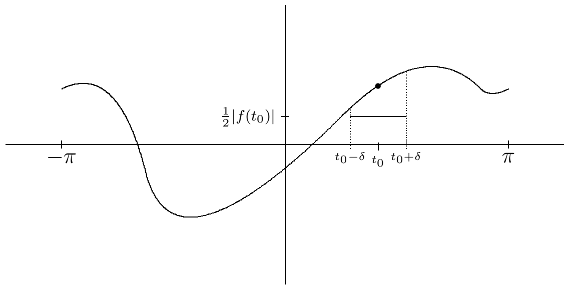
We will let you check properties (1) and (2) for yourself. Let us explain property (3). Firstly

$$\langle f(t), f(t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)^2 dt.$$

As $f(t)^2 \geq 0$ for all t we see that $\langle f(t), f(t) \rangle \geq 0$.
Suppose $f(t) \neq 0$, why is $\langle f(t), f(t) \rangle \neq 0$? Well suppose $f(t_0) \neq 0$. Because f is continuous we can find $\delta > 0$ such that

$$|f(t)| > \frac{1}{2}|f(t_0)| \quad \text{for all } t \in [t_0 - \delta, t_0 + \delta].$$

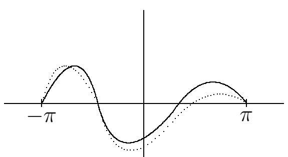
As long as $t_0 \neq \pm\pi$ we may also suppose $t_0 - \delta > -\pi$ and $t_0 + \delta < \pi$. (We leave the cases $t_0 = \pm\pi$ to you, they are only slightly different.)



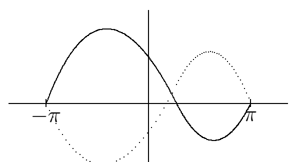
Then

$$\begin{aligned}
\int_{-\pi}^{\pi} f(t)^2 dt &\geq \int_{t_0-\delta}^{t_0+\delta} f(t)^2 dt \\
&\geq \int_{t_0-\delta}^{t_0+\delta} \frac{1}{4}|f(t_0)|^2 dt \\
&\geq \frac{\delta}{2}|f(t_0)|^2 \ > 0.
\end{aligned}$$

We define the <u>length</u> of a function $f \in C[-\pi, \pi]$ to be $\sqrt{\langle f(t), f(t)\rangle}$ and we will denote it $\|f\|$. We define the <u>distance</u> between two functions $f(t), g(t) \in C[-\pi, \pi]$ to be $\|f - g\|$. Roughly speaking two functions $f(t)$ and $g(t)$ are close if the area between their graphs is small.



close functions



far apart functions

<u>Examples</u>

(1) To calculate $\|t\|$

$$\begin{aligned}
\langle t, t\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 dt = \frac{1}{\pi}\left[\frac{t^3}{3}\right]_{-\pi}^{\pi} = \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3} \\
\|t\| &= \sqrt{\frac{2}{3}}\pi
\end{aligned}$$

(2) To calculate the distance between 1 and $|t|$

$$\begin{aligned}
\langle |t|-1, |t|-1\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} (|t|-1)^2 dt = \frac{2}{\pi}\int_{0}^{\pi} (t-1)^2 dt \\
&= \frac{2}{\pi}\int_{0}^{\pi} (t^2 - 2t + 1) dt \\
&= \frac{2}{\pi}\left[\frac{t^3}{3} - t^2 + t\right]_0^{\pi} = \frac{2}{3}\pi^2 - 2\pi + 2 \\
\||t| - 1\| &= \sqrt{\frac{2}{3}\pi^2 - 2\pi + 2}
\end{aligned}$$

(3) If n is a positive integer find $\|\sin nt\|$.

$$\langle \sin nt, \sin nt\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} (\sin nt)^2 dt$$

To evaluate this integral recall the usefull trigonometric formulae:

$$\begin{aligned}
\sin(A + B) &= \sin A \cos B + \cos A \sin B \\
\cos(A + B) &= \cos A \cos B - \sin A \sin B \\
1 &= (\cos A)^2 + (\sin A)^2
\end{aligned}$$

Putting $B = A$ in the second of these we get

$$\begin{aligned}
\cos(2A) &= (\cos A)^2 - (\sin A)^2 \\
&= 1 - 2(\sin A)^2
\end{aligned}$$

Thus

$$\langle \sin nt, \sin nt\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}(1 - \cos 2nt) dt = 1 - \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos 2nt = 1.$$

Thus $\|\sin nt\| = 1$.

Similarly if n is a positive integer one can check that $\|\cos nt\| = 1$. Moreover $\|\frac{1}{\sqrt{2}}\| = 1$.

We will call two functions $f(t), g(t) \in C[-\pi, \pi]$ <u>orthogonal</u> if

$$\langle f(t), g(t)\rangle = 0.$$

We will call a collection of functions $f_1(t), \ldots, f_n(t), \ldots$ (finite or infinite) <u>orthonormal</u> if

(a) $\|f_j(t)\| = 1$ for each j

(b) $\langle f_j(t), f_k(t)\rangle = 0$ if $j \neq k$.

Examples

(1) If $n \neq m$ are positive integers then $\sin nx$ and $\sin mx$ are orthogonal

$$\begin{aligned}\langle \sin nt, \sin mt\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nt \sin mt dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}\big(\cos(n-m)t - \cos(n+m)t\big)dt \\ &= \frac{1}{2\pi}\left[\frac{\sin(n-m)t}{(n-m)} - \frac{\sin(n+m)t}{(n+m)}\right]_{-\pi}^{\pi} = 0.\end{aligned}$$

Again we use the formula for $\cos(A+B)$ (and for $\cos(A-B)$).

(2) In fact the sequence of functions

$$\frac{1}{\sqrt{2}},\ \sin(t),\ \cos(t),\ \sin(2t),\ \cos(2t),\ \sin(3t),\ \cos(3t),\ \ldots$$

is orthonormal. We leave it to you to evaluate the necessary integrals.

The following facts can be proved exactly as they were for $\mathbb{R}^n$.

(1) If $f_1(t), \ldots, f_n(t)$ are orthonormal then they form a basis of an n-dimensional subspace of $\mathbb{R}^n$.

The main point here is to check that $f_1(t), \ldots, f_n(t)$ are linearly independent. Suppose

$$c_1 f_1(t) + \ldots + c_n f_n(t) = 0.$$

Taking the inner product

$$\langle f_j(t), c_1 f_1(t) + \ldots + c_n f_n(t)\rangle = 0$$

we see that

$$0 = c_1\langle f_j(t), f_1(t)\rangle + \cdots + c_n\langle f_j(t), f_n(t)\rangle = c_j$$

for each j.

(2) If $f(t)$ and $g(t)$ are orthogonal then

$$\|f(t)+g(t)\|^2 = \|f(t)\|^2 + \|g(t)\|^2$$

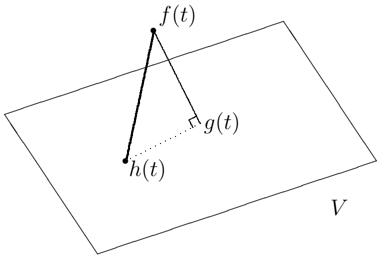
Indeed

$$\begin{aligned}\|f+g\|^2 &= \langle f+g, f+g\rangle \\ &= \langle f,f\rangle + \langle f,g\rangle + \langle g,f\rangle + \langle g,g\rangle \\ &= \langle f,f\rangle + \langle g,g\rangle \\ &= \|f\|^2 + \|g\|^2.\end{aligned}$$

(3) Suppose V is a subspace of $C[-\pi,\pi]$ and that $f(t) \in C[-\pi,\pi]$. If we can find $g(t) \in V$ such that $f(t) - g(t)$ is orthogonal to each element of V then

$$\|f(t) - g(t)\| \leq \|f(t) - h(t)\|$$

for all $h(t) \in V$ with equality if and only if $g(t) = h(t)$.



We have

$$\begin{aligned}\|f(t)-h(t)\|^2 &= \|f(t)-g(t)+g(t)-h(t)\|^2 \\ &= \|f(t)-g(t)\|^2 + \|g(t)-h(t)\|^2 \\ &\geq \|f(t)-g(t)\|^2 \quad \text{with equality only if } \|g(t)-h(t)\| = 0 \text{ i.e. } g(t) = h(t).\end{aligned}$$

The main point is that $g(t) - h(t)$ is in V and so orthogonal to $f(t) - g(t)$.

(4) If $f_1(t), \ldots, f_n(t)$ are an orthonormal basis of a subspace $V \in C[-\pi,\pi]$ then

$$\text{proj}_V\big(f(t)\big) = \langle f(t), f_1(t)\rangle f_1(t) + \cdots + \langle f(t), f_n(t)\rangle f_n(t)$$

is in V; $f(t) - \text{proj}_V\big(f(t)\big)$ is orthogonal to every element of V; and $\text{proj}_V\big(f(t)\big)$ is closer to $f(t)$ then any other element of V.

It suffices to check that for each $j = 1, \ldots, n$: $\langle f_j(t), \text{proj}_V\big(f(t)\big) - f(t)\rangle = 0$

But

$$\begin{aligned}\langle f_j(t), \text{proj}_V\big(f(t)\big) - f(t)\rangle &= \langle f(t), f_1(t)\rangle\langle f_j(t), f_1(t)\rangle + \cdots \\ &\quad + \langle f(t), f_n(t)\rangle\langle f_j(t), f_n(t)\rangle - \langle f_j(t), f(t)\rangle \\ &= \langle f(t), f_j(t)\rangle - \langle f_j(t), f(t)\rangle = 0.\end{aligned}$$

We will let T_n denote the subspace of $C[-\pi,\pi]$ with orthonormal basis

$$\frac{1}{\sqrt{2}}\ ,\ \sin t\ ,\ \cos t\ ,\ \ldots\ ,\ \sin nt\ ,\ \cos nt.$$

Then $\text{proj}_{T_n}\big(f(t)\big)$ is an approximation to $f(t)$ constructed from these trigonometric functions. As n increases one might expect these approximations to become better and better.

In fact we have:

Fact 10.2.1. *(1) If $f \in C[-\pi,\pi]$ then*

$$\|\mathrm{proj}_{T_n}\big(f(t)\big) - f(t)\| \to 0$$

as $n \to \infty$.

(2)

$$\|f(t)\|^2 = \langle f(t), \frac{1}{\sqrt{2}}\rangle^2 + \sum_1^\infty \Big(\langle f(t), \sin nt\rangle^2 + \langle f(t), \cos nt\rangle^2\Big)$$

Although we will not prove this, let us at least explain how (2) follows from (1).

$$\begin{aligned} \big\|\mathrm{proj}_{T_n}\big(f(t)\big)\big\|^2 &= \left\|\langle f(t), \frac{1}{\sqrt{2}}\rangle\frac{1}{\sqrt{2}} + \langle f(t), \sin t\rangle \sin t + \cdots + \langle f(t), \cos nt\rangle \cos nt\right\|^2 \\ &= \langle f(t), \frac{1}{\sqrt{2}}\rangle^2 + \langle f(t), \sin nt\rangle^2 + \cdots + \langle f(t), \cos nt\rangle^2. \end{aligned}$$

On the other hand

$$\begin{aligned} \|f(t)\|^2 &= \|\mathrm{proj}_{T_n}\big(f(t)\big)\|^2 + \|f(t) - \mathrm{proj}_{T_n}\big(f(t)\big)\|^2 \\ &= \langle f(t), \frac{1}{\sqrt{2}}\rangle^2 + \cdots + \langle f(t), \cos nt\rangle^2 + \|f(t) - \mathrm{proj}_{T_n}\big(f(t)\big)\|^2. \end{aligned}$$

Letting $n \to \infty$ gives part (2).

Although this tells us that "on average" $\mathrm{proj}_{T_n}\big(f(t)\big)$ is close to $f(t)$, it does not tell us what happens for any given $t \in [-\pi,\pi]$. However if we place some smoothness hypothesis on $f(t)$ then we can say what happens.

Fact 10.2.2. *Suppose $f(t) \in C[-\pi,\pi]$ is differentiable at a point $x \in [-\pi,\pi]$, and if $x = \pm\pi$ also assume that $f(-\pi) = f(\pi)$. Then the series*

$$\langle \frac{1}{\sqrt{2}}, f(t)\rangle \frac{1}{\sqrt{2}} + \sum_{n=1}^\infty \Big(\langle \sin nt, f(t)\rangle \sin nx + \langle \cos nt, f(t)\rangle \cos nx\Big)$$

converges to $f(x)$.

The series

$$\langle \frac{1}{\sqrt{2}}, f(t)\rangle \frac{1}{\sqrt{2}} + \sum_{n=1}^\infty \Big(\langle \sin nt, f(t)\rangle \sin nt + \langle \cos nt, f(t)\rangle \cos nt\Big)$$

is called the Fourier series for f after the French mathematician Jean-Baptiste-Joseph Fourier (1768-1830). Fact 10.2.2 was known to Fourier and is often referred to as Fourier's theorem, although the first rigorous proof was only found later by Dirichlet.

We recommend that you read section 5.5.

Example Find the Fourier series for t.

$$\begin{aligned} \langle \frac{1}{\sqrt{2}}, t\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{t}{\sqrt{2}}dt = 0 \\ \langle \cos nt, t\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos nt dt = 0 \quad \text{because } t\cos nt \text{ is an odd function} \\ \langle \sin nt, t\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin nt dt \\ &= \frac{1}{\pi}\left[\frac{-\cos nt}{n}\, t\right]_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos nt}{n}dt \\ &= \frac{1}{\pi}\left(\frac{-(-1)^n}{n}\,\pi - \frac{-(-1)^n}{n}\,(-\pi)\right) \\ &= \frac{2(-1)^{n+1}}{n} \end{aligned}$$

Thus

$$t = 2\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}\sin nt.$$

By part (2) of Fact 10.2.1 we see that

$$\|t\|^2 = \sum_{n=1}^\infty \frac{4}{n^2}$$

i.e.

$$\frac{2}{3}\pi^2 = 4\sum_{n=1}^\infty \frac{1}{n^2} \qquad \text{i.e.} \quad \frac{\pi^2}{6} = \sum_{n=1}^\infty \frac{1}{n^2}$$

i.e.

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots$$

an amazing expression of π as an infinite sum.

On the other hand by Fact 10.2.2 if we put $t = \frac{\pi}{2}$ we get

$$\frac{\pi}{2} = 2\sum_{\substack{n=1 \\ n \text{ odd}}}^\infty \frac{(-1)^{n+1}}{n}\,(-1)^{\frac{n-1}{2}}$$

i.e.

$$\frac{\pi}{4} = \sum_{m=0}^\infty \frac{(-1)^m}{2m+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots,$$

another amazing expression for π as an infinite sum.

EXERCISES

(1) Find the length of
$$1 + \sin t + 3\cos 5t + 2\sin 10t$$

(2) Show that $1/\sqrt{2}$ and $\sqrt{3/2}\, t/\pi$ are orthonormal. Let V be the subspace of $C[-\pi, \pi]$ consisting of functions of the form $at + b$. Find $\mathrm{proj}_V(t^2)$.

(3) Find the Fourier series for $|t|$.

(4) Calculate $\displaystyle\int_{-\pi}^{\pi} e^{at}\cos nt dt$.

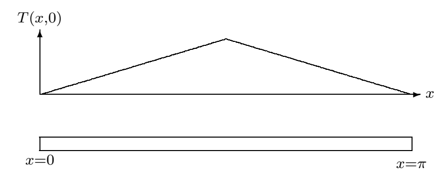
[HINT: Integrate by parts twice to get an expression $\int_{-\pi}^{\pi} e^{at}\cos nt dt = \left[\frac{a\cos nt}{n^2}e^{at}\right]_{-\pi}^{\pi} - \frac{a^2}{n^2}\int_{-\pi}^{\pi} e^{at}\cos nt dt$ and then solve for $\int_{-\pi}^{\pi} e^{at}\cos nt dt$]

(5) If a is a real constant find the Fourier series for
$$\cosh at = \frac{1}{2}\left(e^{at} + e^{-at}\right)$$
[HINT: $\cosh(-at) = \cosh(at)$]

(6) Find a closed formula for $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}$ as a function of a. [HINT: use (5) and Fact 10.2.2]

10.3 Partial Differential Equation I: The Heat Equation

Consider a uniform metal bar stretching from $x = 0$ to $x = \pi$. Suppose that the ends of the bar are held at a constant temperature of 0 (eg. are immersed in a mixture of water and ice) but that otherwise the bar is thermally insulated from its surroundings, except that at time $t = 0$ the bar is quickly heated so that it has temperature distribution
$$T(x,0) = \begin{cases} x & \text{if } x \le \frac{\pi}{2} \\ \pi - x & \text{if } x \ge \frac{\pi}{2} \end{cases}$$
Describe the temperature of the bar at all subsequent times.



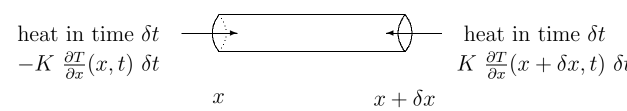
The temperature $T(x,t)$ obeys the equation:
$$\frac{\partial T}{\partial t} = \mu\frac{\partial^2 T}{\partial x^2}$$
for some positive constant μ depending on the structure of the bar. (This sort of equation is called a partial differential equation or PDE)

Where does this particular equation come from?

rate of heat flow past x is $-K\frac{\partial T}{\partial x}$, K = thermal conductivity. (heat flows from hot to cold at a rate proportional to the temperature gradient)

rate of temperature increase = C . rate of arrival of heat (C = heat capacity)

We examine what happens to a small length of bar from x to $x + \delta x$ in the small time from t to $t + \delta t$.



total heat in time δt :
$$K\left(\frac{\partial T}{\partial x}(x+\delta x, t) - \frac{\partial T}{\partial x}(x,t)\right)\delta t$$
rise in temperature in time δt :
$$T(x, t+\delta t) - T(x,t) \approx \frac{C}{\delta x}K\left(\frac{\partial T}{\partial x}(x+\delta x, t) - \frac{\partial T}{\partial x}(x,t)\right)\delta t$$

i.e.

$$\frac{1}{\delta t}\big(T(x,t+\delta t)-T(x,t)\big)\approx CK\frac{1}{\delta x}\left(\frac{\partial T}{\partial x}(x+\delta x,t)-\frac{\partial T}{\partial x}(x,t)\right)$$

i.e.

$$\frac{\partial T}{\partial t}=CK\frac{\partial^2 T}{\partial x^2}$$

The equation

$$\boxed{\frac{\partial T}{\partial t}=\mu\frac{\partial^2 T}{\partial x^2}}$$

is called the heat or diffusion equation. It arises in many physical situations where some diffusion process occurs eg. diffusion of pollutants in an aquifer, or of ions through a cell wall.

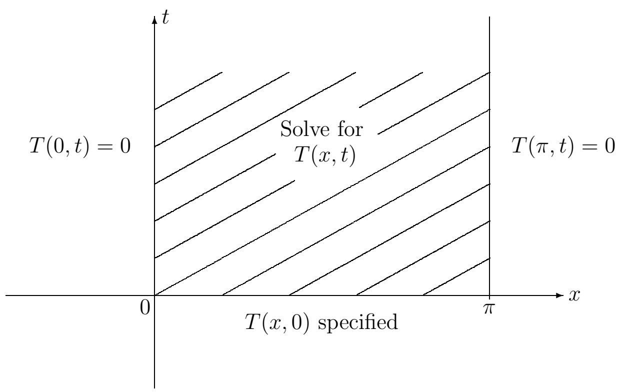
Here we are asked to find a solution to this equation subject to the restrictions that

$$\begin{aligned} T(0,t) &= T(\pi,t)=0 \quad \text{(the ends of the bar stay at temperature 0)}\\ T(x,0) &= \begin{cases} x & \text{if } x\le\frac{\pi}{2}\\ \pi-x & \text{if } x>\frac{\pi}{2}.\end{cases}\end{aligned}$$

Such restrictions are called initial conditions or boundary conditions. Many different initial conditions are possible, they will depend on the problem one is trying to solve. (Another possibility would be that the bar was initially at temperature 0, that the left end is always kept at temperature 0 but that the right end is made to take on a specified temperature $T(\pi,t)$.)

We are only looking for T in the region

$$\begin{aligned} 0\le x\le\pi &\quad \text{(length of bar)}\\ t\ge 0 &\quad \text{(positive time)}\end{aligned}$$



There are several methods available to tackle this sort of problem, we will present one based on Fourier series.

We first look for some simple solutions to the equation

$$\frac{\partial T}{\partial t}=\mu\frac{\partial^2 T}{\partial x^2}\qquad T(0,t)=T(\pi,t)=0. \tag{$*$}$$

In fact let us look for a solution

$$T(x,t)=u(x)v(t).$$

Then we require $u(0)=u(\pi)=0$ and

$$\frac{v'(t)}{v(t)}=\mu\frac{u''(x)}{u(x)}.$$

We see that the quantity

$$\frac{v'(t)}{v(t)}=\mu\frac{u''(x)}{u(x)}$$

is independent of both position x and time t so that it must be a constant.
We are led to try to solve the equation

$$\begin{aligned} u''(x) &= au(x)\\ v'(t) &= a\mu v(t)\\ u(0) &= u(\pi)=0.\end{aligned}$$

But we know how to solve these equation.

(a) If $a>0$, say $a=\lambda^2$ then

$$u(x)=Ae^{\lambda x}+Be^{-\lambda x}.$$

The equation $u(0)=u(\pi)=0$ imply that $A=B=0$, i.e. $u(x)\equiv 0$. This is not much help.

(b) If $a=0$ then

$$u(x)=Ax+B.$$

Again the equation $u(0)=u(\pi)=0$ imply that $A=B=0$, i.e. $u(x)\equiv 0$. Again not much help.

(c) Now suppose $a<0$, say $a=-\lambda^2$. Then

$$u(x)=A\sin\lambda x+B\cos\lambda x.$$

The equation $u(0)=0$ implies $B=0$.

The equation $u(\pi)=0$ implies $A=0$ or λ is a whole number n. In this case $v'(t)=-n^2\mu v(t)$ so that $v(t)=Ce^{-\mu n^2 t}$.

Thus we have found a series of solutions to $(*)$. Namely for each positive integer n we have a solution

$$c_n e^{-\mu n^2 t}\sin nx$$

for any constant c_n. If we put $t=0$ we get $c_n \sin nx$ so none of these solutions is the one we are looking for.
However note that both the equations

$$\frac{\partial T}{\partial t}=\mu\frac{\partial^2 T}{\partial x^2}$$

and the boundary conditions

$$T(0,t)=T(\pi,t)=0$$

are <u>linear</u>: i.e. if T_1 and T_2 are two solutions so is cT_1+T_2. Thus we get a lot more solutions of these two equations: namely any finite sum

$$\sum_{n=1}^{N} c_n e^{-\mu n^2 t}\sin nx.$$

At $t=0$ this becomes

$$\sum_{n=1}^{N} c_n \sin nx.$$

In fact more is true. If the constants c_n become smaller sufficiently rapidly as $n\to\infty$ then the sum

$$\sum_{n=1}^{\infty} c_n e^{-\mu n^2 t}\sin nx$$

will converge and give a solution to $(*)$ which specialises at $t=0$ to

$$\sum_{n=1}^{\infty} c_n \sin nx.$$

If we can find c_n such that

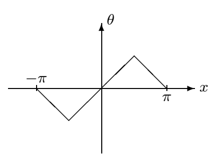
$$\sum_{n=1}^{\infty} c_n \sin nx=\begin{cases} x & x\le\frac{\pi}{2}\\ \pi-x & x\ge\frac{\pi}{2}\end{cases}$$

then we would have found a solution to our original problem. But this is the sort of problem we studied in the last section.

To put it more precisely in the form we considered in the last section consider

$$\theta(x)=\begin{cases}\pi-x & x\ge\frac{\pi}{2}\\ x & -\frac{\pi}{2}\le x\le\frac{\pi}{2}\\ -x-\pi & x\le-\frac{\pi}{2}\end{cases}$$

Note that we extended θ to $[-\pi,\pi]$ by arranging that $\theta(-x)=-\theta(x)$.



We now compute the Fourier series of θ.

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\theta(x)\frac{1}{\sqrt{2}}dx=0 \qquad \text{as } \theta(x)=-\theta(-x).$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\theta(x)\cos nx dx=0 \qquad \text{for the same reason.}$$

$$\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi}\theta(x)\sin nxdx &= \frac{2}{\pi}\int_0^{\pi}\theta(x)\sin nxdx\\
&= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\sin nxdx+\frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}(\pi-x)\sin nxdx\\
&= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\sin nxdx-\frac{2}{\pi}\int_{\frac{\pi}{2}}^{0} y\sin n(\pi-y)dy\\
&= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\sin nxdx-\frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} y\sin(ny-n\pi)dy\\
&= \frac{2}{\pi}\left(1-(-1)^n\right)\int_0^{\frac{\pi}{2}} x\sin nxdx\\
&= 0 \qquad\qquad n \text{ even}\\
&= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} x\sin nxdx\\
&= \frac{4}{\pi}\left[\frac{-\cos nx}{n}x\right]_0^{\frac{\pi}{2}}+\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\frac{\cos nx}{n}dx\\
&= \frac{4}{\pi n^2}\left[\sin nx\right]_0^{\frac{\pi}{2}}\\
&= \frac{4}{n^2\pi}(-1)^{\frac{n-1}{2}} \qquad n \text{ odd}
\end{aligned}$$

$$\theta(x)=\sum_{m=0}^{\infty}\frac{4\,(-1)^m}{(2m+1)^2\pi}\sin(2m+1)x$$

Thus we see that

$$T(x,t)=\sum_{m=0}^{\infty}\frac{4\,(-1)^m}{(2m+1)^2\pi}e^{-\mu(2m+1)^2 t}\sin\,(2m+1)x$$

satisfies

$$\begin{aligned}
&\frac{\partial T}{\partial t}=-\mu\frac{\partial^2 T}{\partial x^2}\\
&T(0,t)=T(\pi,t)=0\\
&T(x,0)=\theta(x)
\end{aligned}$$

as desired.

Note that as $t \to \infty$, $e^{-\mu(2m+1)^2 t} \to 0$. Thus as $t \to \infty$, $T(x,t) \to 0$. As one might have expected the bar cools towards having a uniform temperature of 0.
The same method (developed by Fourier at the start of the 19th century) allows one to so solve the heat equation with any boundary conditions of this form. In fact we have:

FACT Let $f(x)$ be any (reasonable) function on $[0,\pi]$ which vanishes at both end points. Then there is a unique function $T(x,t)$ for $0 \le x \le \pi$, $t \ge 0$ such that

$$\frac{\partial T}{\partial t} = -\mu \frac{\partial^2 T}{\partial x^2}$$
$$T(0,t) = T(\pi,t) = 0$$
$$T(x,0) = f(x).$$

EXERCISES

(1) Solve the equation $\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2}$ in $0 \le x \le \pi$, $t \ge 0$ subject to $T(0,t) = T(\pi,t)$ and $T(x,0) = 4 \sin x$.

(2) Solve the equation $\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2}$ in $0 \le x \le \pi$, $t \ge 0$ subject to $T(0,t) = T(\pi,t) = 0$ and $T(x,0) = \begin{cases} 0 & x \le \pi/4 \\ 1 & \pi/4 < x < 3\pi/4 \\ 0 & x \ge 3\pi/4 \end{cases}$

[You may assume that $T(x,0)$ has a Fourier sine series, which can be computed in the same way as when T is continuous.]

(3) Show that $T(x,t) = \frac{100}{\pi} x$ is a solution of $\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$ subject to $T(0,t)$, $T(\pi,t) = 100$.

(4) Solve the equation $\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$ in $0 \le x \le \pi$, $t \ge 0$ subject to $T(0,t) = 0$, $T(\pi,t) = 100$, $T(x,0) = 0$ for $0 \le x \le \pi$. Describe $T(x,t)$ for very large t. [HINT: look for a solution $T(x,t) = \frac{100}{\pi} x + S(x,t)$.]

(5) Show that if $n = 0, 1, 2, 3, \dots$ then

$$T(x,t) = e^{-n^2 \mu t} \cos nx$$

is a solution of $\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$ such that $\frac{\partial T}{\partial x}(0,t) = \frac{\partial T}{\partial x}(\pi,t) = 0$. (These boundary conditions correspond to a bar which is completely thermally insulated, even at its ends.)

(6) Solve the equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \qquad \text{in } 0 \le x \le \pi,\ t \ge 0$$

and subject to the boundary conditions $\frac{\partial T}{\partial x}(0,t) = \frac{\partial T}{\partial x}(\pi,t) = 0$ and $T(x,0) = x$. Describe $T(x,t)$ for very large t.

10.4 Partial Differential Equations II

We will discuss two other very standard examples of PDE's.

1) Laplace's Equation

Consider a square copper plate: $0 \le x \le \pi$, $0 \le y \le \pi$. The sides $y = 0$, $y = \pi$ and $x = 0$ are maintained at a constant temperature of 0. The point (π, y) is maintained at a temperature

$$\begin{array}{rll} y & \text{if} & 0 \le y \le \pi/2 \\ \pi - y & \text{if} & \pi/2 \le y \le \pi \end{array}$$

If the plate is in equilibrium find the temperature distribution on the plate.

The temperature $T(x, y, t)$ satisfies

$$\frac{\partial T}{\partial t} = \mu \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right).$$

If the temperature is constant then we must have

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

This is called Laplace's equation.
We must solve Laplace's equation in $0 \le x \le \pi$, $0 \le y \le \pi$ subject to $T(x, 0) = T(x, \pi) = 0$, $T(0, y) = 0$,

$$T(\pi, y) = \begin{cases} y & y \le \pi/2 \\ \pi - y & y \le \pi/2 \end{cases}$$

Again we look for simple solutions to Laplace's equation of the form

$$T(x, y) = u(x)v(y)$$

We must then solve

$$\begin{array}{ll} u''(x) = au(x) & u(0) = 0 \\ v''(y) = -av(y) & v(0) = v(\pi) = 0 \end{array}$$

As for the heat equation the only non-trivial solutions are for $a = n^2$; $n = 1, 2, 3, \ldots$
Then

$$\begin{aligned} v(y) &= A \sin ny \\ u(x) &= B\left(e^{nx} - e^{-nx}\right) = 2B \sinh(nx) \end{aligned}$$

Thus we get the solutions

$$c_n = \sinh(nx) \sin(ny)$$

By linearity

$$\sum_{n=1}^{\infty} c_n \sinh(nx) \sin(ny)$$

will also be a solution if c_n tend to zero sufficiently fast.

We would like to choose c_n such that

$$\sum_{n=1}^{\infty} c_n \sinh(n\pi) \sin(ny) = \begin{cases} y & y \le \pi/2 \\ \pi - y & y \ge \pi/2 \end{cases}$$

As in the last section we see that

$$c_n \sinh(n\pi) = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n}\frac{(-1)^{\frac{n-1}{2}}}{\pi} & n \text{ odd} \end{cases}$$

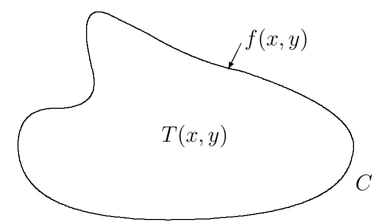
Thus

$$T(x, y) = \sum_{m=0}^{\infty} \frac{4\,(-1)^m}{\pi(2m+1)} \frac{\sinh\big((2m+1)x\big)}{\sinh\big((2m+1)\pi\big)} \sin\big((2m+1)y\big).$$

<u>FACT</u> If C is any smooth simple closed curve in the plane and f is a smooth function on C then we can find a function T on the interior of C such that

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

and for $(x, y) \in C$: $T(x, y) = f(x, y)$.



2) The Wave Equation

Suppose a violin string of length π is fixed between the points $x = 0$ and $x = \pi$ and suppose the string is plucked with the end points fixed. Describe the movement of the string.

Let $u(x, t)$ denote the displacement of the string from the x-axis at time t and at distance x along the x-axis.

Then u satisfies the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

(as long as time is measures in suitable units).
We are looking for solutions satisfying the boundary conditions $u(0,t) = u(\pi,t) = 0$.

We look again for simple solutions

$$u(x,t) = v(x)w(t)$$

and obtain the equations

$$\begin{aligned} v''(x) &= av(x) \qquad v(x) = v(\pi) = 0 \\ w''(t) &= aw(t) \end{aligned}$$

As in the previous section we see we only obtain a non-trivial solution if $a = -n^2$ for n an integer.
Thus we obtain solutions

$$a_n \sin nt \sin nx \qquad \text{and} \qquad b_n \cos nt \sin nx.$$

Again linearity gives solutions

$$\sum \big(a_n \sin nt + b_n \cos nt\big) \sin nx$$

where we may in fact allow the sums to become infinite if a_n and b_n tend to zero sufficiently fast as $n \to \infty$.
To get a specific solution we must specify what happens at $t = 0$. Suppose that at $t = 0$ the string is stationary with

$$u(x,0) = \begin{cases} \frac{x}{100} & x \leq \pi/2 \\ \frac{\pi - x}{100} & x \geq \pi/2. \end{cases}$$

Then we rquire that

$$\sum_{n=1}^{\infty} b_n \sin nx = \begin{cases} \frac{x}{100} & x \leq \pi/2 \\ \frac{\pi - x}{100} & x \geq \pi/2. \end{cases}$$

As in the last section we see that $b_n = \begin{cases} 0 & n \text{ even} \\ \frac{4}{100n^2\pi}(-1)^{\frac{n-1}{2}} & n \text{ odd.} \end{cases}$

What about the a_n ? They seem to be arbitrary. The point is that the motion of the string depends not only on its initial position, but also on its initial velocity. Using the fact that the string is stationary at $t = 0$ we see that

$$\sum \Big(na_n \cos nt - nb_n \sin nt\Big) \sin nx\Big|_{t=0} = 0.$$

Thus $\sum na_n \sin nx = 0$ and so $a_n = 0$ for all n. Thus

$$u(x,t) = \sum_{m=0}^{\infty} \frac{4\ (-1)^m}{100(2m+1)^2\pi} \cos(2m+1)t \sin(2m+1)x.$$

Notice the difference. The equation $\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$ has a unique solution in $0 \leq x \leq \pi$, $t \geq 0$ if we specify $T(0,t) = T(\pi,t) = 0$ and we specify $T(x,0)$.
On the other hand the equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ has infinitely many solutions in $0 \leq x \leq \pi$, $t \geq 0$ is we specify $u(0,t) = u(\pi,t) = 0$ and we specify $u(x,0)$. In this case we may also specify $\frac{\partial u}{\partial t}(x,0)$.
In general it is a subtle question what boundary conditions we can impose for a PDE and still expect a solution or a unique solution.

EXERCISES

(1) Suppose that the the boundary of a uniform copper disc is maintained at a temperature $T(x, y) = xy$.

Find the temperature at the center of the disc when the temperature over the disc is constant in time.

(2) A uniform metal square $0 \le x \le \pi$, $0 \le y \le \pi$ has a temperature distribution which is constant in time. If

$$T(0, y) = 0 \qquad T(\pi, y) = \begin{cases} y & y \le \pi/2 \\ \pi - y & y \ge \pi/2 \end{cases}$$

$$T(x, 0) = 0 \qquad T(x, \pi) = \begin{cases} x & x \le \pi/2 \\ \pi - x & x \ge \pi/2 \end{cases}$$

find $T(x, y)$ over the whole square.

(3) A violin string fixed at $x = 0$ and $x = \pi$ is initially undisturbed.

It is then given a velocity

$$\frac{\partial u}{\partial t}(x, 0) = \begin{cases} x & x \le \pi/2 \\ \pi - x & x \ge \pi/2. \end{cases}$$

Describe the displacement $u(x, t)$ of the string as a function of position and time.

(4) Show that if $f(y)$ and $g(y)$ are any twice differentiable functions then $u(x, t) = f(x + t) + g(x - t)$ satisfies

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}.$$

If $u(0, t) = u(\pi, t) = 0$ show that we must have

$$f(y) = -g(-y)$$
$$f(y + 2\pi) = f(y)$$
$$u(x, t) = f(x + t) - f(t - x).$$

If further $\dfrac{\partial u}{\partial t}(x, 0) = 0$ show that $f(y) + f(-y)$ is constant and hence that $f(y) = -f(-y)$.

If $u(x, 0) = \begin{cases} x & x \le \pi/2 \\ \pi - x & x \ge \pi/2 \end{cases}$ find f and hence find $u(x, t)$.

(5) Solve the equation $\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} = 0$ in $0 \le x \le \pi$, $0 \le y \le \pi$ subject to

$$\frac{\partial T}{\partial x}(0, y) = 0 \ , \quad \frac{\partial T}{\partial x}(\pi, y) = \sin 2y$$
$$\frac{\partial T}{\partial y}(x, 0) = 0 \ , \quad \frac{\partial T}{\partial y}(x, \pi) = 0$$

(6) Can you solve the equation $\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} = 0$ in $0 \le x \le \pi$, $0 \le y \le \pi$ subject to

$$\frac{\partial T}{\partial x}(0, y) = 0 \ , \quad \frac{\partial T}{\partial x}(\pi, y) = \sin y$$
$$\frac{\partial T}{\partial y}(x, 0) = 0 \ , \quad \frac{\partial T}{\partial y}(x, \pi) = 0$$

[HINT: Apply Green's theorem to $\left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}\right)$.]