Symplectic Toric Manifolds

Ana Cannas da Silva[1]

July 2001

[1]E-mail: acannas@math.ist.utl.pt

Foreword

These notes cover a short course on symplectic toric manifolds, delivered in six lectures at the summer school on Symplectic Geometry of Integrable Hamiltonian Systems, mostly for graduate students, held at the Centre de Recerca Matemàtica in Barcelona in July of 2001.

The goal of this course is to provide a fast elementary introduction to toric manifolds (i.e., smooth toric varieties) from the symplectic viewpoint. The study of toric manifolds has many different entrances and has been scoring a wide spectrum of applications. For symplectic geometers, they provide examples of extremely symmetric and completely integrable hamiltonian spaces. In order to distinguish the algebraic from the symplectic approach, we call *algebraic toric manifolds* to the smooth toric varieties in algebraic geometry, and say *symplectic toric manifolds* when studying their symplectic properties.

Native to algebraic geometry, the theory of toric varieties has been around for about thirty years. It was introduced by Demazure in [16] who used toric varieties for classifying some algebraic subgroups. Since 1970 many nice surveys of the theory of toric varieties have appeared (see, for instance, [14, 21, 28, 41]). Algebraic geometers and combinatorialists have found fruitful applications of toric varieties to the geometry of convex polytopes, resolutions of singularities, compactifications of locally symmetric spaces, critical points of analytic functions, etc. For the last ten years, toric geometry became an important tool in physics in connection with mirror symmetry [13] where research has been intensive.

In this text we emphasize the geometry of the *moment map* whose image, the so-called *moment polytope*, determines the symplectic toric manifolds. The notion of a moment map associated to a group action generalizes that of a hamiltonian function associated to a vector field. Either of these notions formalizes the Noether principle, which states that to every symmetry (such as a group action) in a mechanical system, there corresponds a conserved quantity. The concept of a moment map was introduced by Souriau [45] under the french name *application moment*; besides the more standard english translation to *moment map*, the alternative *momentum map* is also used. Moment maps have been asserting themselves as a main tool to study problems in geometry and topology when there is a suitable symmetry, as illustrated in the book by Gelfand, Kapranov and Zelevinsky [22]. The material in some sections of the second part of these notes borrows largely from that excellent text, where details are given and where the discussion continues in exciting new directions.

This course splits into two parts: the first concentrates on the *symplectic viewpoint*, and the second focuses on the *algebraic viewpoint* with links to symplectic geometry. Each of the two parts has a similar structure: introduction, classification, polytopes. So the lectures seem periodic though the languages for each half are noticeably different.

After introducing, in the first lecture, basic notions related to symplectic toric manifolds, in Lecture 2 we state their classification, and prove the existence part of the classification theorem by using the technique of symplectic reduction. Lecture 3 discusses moment polytopes, namely how to read from the polytope some topological properties of the corresponding symplectic toric manifolds, as well as how are some changes on the polytopes translated into changes of the corresponding symplectic toric manifolds. Lecture 4 introduces toric manifolds from the algebro-geometric perspective, after reviewing definitions and notation in the theory of algebraic varieties. Lecture 5 describes the classification of toric varieties using the language of spectra and fans. Finally, Lecture 6 deals with polytopes now from the algebraic point of view, studying some geometric properties of toric varieties which polytopes encode. Lectures 3 and 6 underline the moment map potential.

Geometry of manifolds at the level of a first-year graduate course is the basic prerequisite for this course. Some familiarity with symplectic geometry is useful to read these notes faster, though most of the needed definitions and results are stated here. Scattered through the text, there are exercises designed to complement the exposition or extend the reader's understanding. Throughout, the symbol $\mathbb{P}^n$ denotes n-(complex-)dimensional complex projective space.

Many thanks to Carlos Curràs-Bosch and to Eva Miranda for the invitation!

Ana Cannas da Silva

Strasbourg, June 2001
Barcelona, July 2001
Bonn, August 2001
Lisbon, September 2001
Princeton, October 2001

Contents

Part I

Symplectic Viewpoint

Chapter 1

Symplectic Toric Manifolds

In order to define symplectic toric manifolds, we begin by introducing the basic objects in symplectic/hamiltonian geometry/mechanics which lead to their consideration. Our discussion centers around moment maps.

1.1 Symplectic Manifolds

Definition 1.1.1. *A* **symplectic form** *on a manifold M is a closed 2-form on M which is nondegenerate at every point of M. A* **symplectic manifold** *is a pair (M,ω) where M is a manifold and ω is a symplectic form on M.*

By linear algebra, a symplectic manifold is necessarily even-dimensional.

Examples.

1. Let $M = \mathbb{R}^{2n}$ with linear coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$. The **standard symplectic form on** $\mathbb{R}^{2n}$ is

$$\omega_0 = \sum_{k=1}^{n} dx_k \wedge dy_k \ .$$

2. Let $M = \mathbb{C}^n$ with linear coordinates $z_1, \ldots, z_n$. The form

$$\omega_0 = \frac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k$$

is a symplectic form on $\mathbb{C}^n$. In fact, this form equals that of the previous example under the identification $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, $z_k = x_k + iy_k$.

3. Let $M = S^2$ regarded as the set of unit vectors in $\mathbb{R}^3$. Tangent vectors to S^2 at p may then be identified with vectors orthogonal to p. The **standard symplectic form on** S^2 is the form induced by the inner and exterior products:

$$\omega_p(u,v) := \langle p, u \times v \rangle \ , \qquad \text{for } u,v \in T_pS^2 = \{p\}^\perp \ .$$

This form is closed because it is of top degree; it is nondegenerate because $\langle p, u \times v \rangle \neq 0$ when $u \neq 0$ and we take, for instance, $v = u \times p$.

$\diamondsuit$

Exercise 1
Check that, in cylindrical coordinates away from the poles ($0 \leq \theta < 2\pi$ and $-1 < h < 1$), the standard symplectic form on S^2 is the area form given by

$$\omega_{\text{standard}} = d\theta \wedge dh \ .$$

The natural notion of equivalence in the symplectic category is expressed by a *symplectomorphism*:

Definition 1.1.2. *Let (M_1, ω_1) and (M_2, ω_2) be $2n$-dimensional symplectic manifolds, and let $\varphi : M_1 \to M_2$ be a diffeomorphism. Then φ is a* **symplectomorphism** *if $\varphi^*\omega_2 = \omega_1$.*

The Darboux theorem (see, for instance, [12] or Theorem 3.1.1 for the case where the group is trivial) states that any symplectic manifold (M^{2n}, ω) is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$. In other words, the prototype of a local piece of a $2n$-dimensional symplectic manifold is $(\mathbb{R}^{2n}, \omega_0)$. Hence, this theorem provides the local classification of symplectic manifolds in terms of a unique invariant: the dimension.

Let (M, ω) be a symplectic manifold of dimension $2n$. A **Darboux chart** for M is a chart $(\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots, y_n)$ such that

$$\omega|_{\mathcal{U}} = \sum_{k=1}^{n} dx_k \wedge dy_k \ .$$

By the Darboux theorem, there exists a Darboux chart centered at each point of a symplectic manifold.

1.2 Hamiltonian Vector Fields

Let (M, ω) be a symplectic manifold.

Definition 1.2.1. *A vector field X on M is* **symplectic** *if the contraction $\imath_X\omega$ is closed. A vector field X on M is* **hamiltonian** *if the contraction $\imath_X\omega$ is exact.*

Locally on every contractible open set, every symplectic vector field is hamiltonian. If the first de Rham cohomology group is trivial, then globally every symplectic vector field is hamiltonian; in general, $H^1_{\text{deRham}}(M)$ measures the obstruction for symplectic vector fields to be hamiltonian.

Note that the flow of a symplectic vector field X preserves the symplectic form:

$$\mathcal{L}_X\omega = d\underbrace{\imath_X\omega}_{\text{closed}} + \imath_X\underbrace{d\omega}_{0} = 0\ .$$

If a vector field X is hamiltonian with $\imath_X\omega = dH$ for some smooth function $H : M \to \mathbb{R}$, then the flow of X also preserves the function H:

$$\mathcal{L}_X H = \imath_X dH = \imath_X\, \imath_X\omega = 0\ .$$

Therefore, each integral curve $\{\rho_t(x) \mid t \in \mathbb{R}\}$ of X must be contained in a level set of H:

$$H(x) = (\rho_t^* H)(x) = H(\rho_t(x))\ , \quad \forall t\ .$$

Definition 1.2.2. *A* **hamiltonian function** *for a hamiltonian vector field X on M is a smooth function $H : M \to \mathbb{R}$ such that $\imath_X\omega = dH$.*

By nondegeneracy of ω, any function $H \in C^\infty(M)$ is a hamiltonian function for some hamiltonian vector field because the equation $\imath_X\omega = dH$ can be always solved for a smooth vector field X. A hamiltonian vector field X defines a hamiltonian function *up to* a locally constant function.

Examples.

1. On the standard symplectic 2-sphere $(S^2, d\theta \wedge dh)$, the vector field $X = \frac{\partial}{\partial\theta}$ is hamiltonian with hamiltonian function given by the height function:

   $$\imath_X(d\theta \wedge dh) = dh\ .$$

   The motion generated by this vector field is rotation about the vertical axis, which of course preserves both area and height.

2. On the symplectic 2-torus $(\mathbb{T}^2, d\theta_1 \wedge d\theta_2)$, the vector fields $X_1 = \frac{\partial}{\partial\theta_1}$ and $X_2 = \frac{\partial}{\partial\theta_2}$ are symplectic but not hamiltonian.

$\diamondsuit$

1.3 Integrable Systems

Let X_H denote a hamiltonian vector field on a symplectic manifold (M, ω) with hamiltonian function $H \in C^\infty(M)$.

Definition 1.3.1. *The* **Poisson bracket** *of two functions $f, g \in C^\infty(M)$ is the function*

$$\{f, g\} := \omega(X_f, X_g) \ .$$

We have $X_{\{f,g\}} = -[X_f, X_g]$ because $X_{\omega(X_f,X_g)} = [X_g, X_f]$, where $[\cdot, \cdot]$ is the Lie bracket of vector fields.[1]

Exercise 2
Check that the Poisson bracket $\{\cdot, \cdot\}$ is a Lie bracket and satisfies the **Leibniz rule**:

$$\{f, gh\} = \{f, g\}h + g\{f, h\} \qquad \forall f, g, h \in C^\infty(M) \ .$$

Therefore, if (M, ω) is a symplectic manifold, then $(C^\infty(M), \{\cdot, \cdot\})$ is a Poisson algebra, that is a Lie algebra with an associative product for which the Leibniz rule holds. Furthermore, we have an anti-homomorphism of Lie algebras

$$\begin{array}{ccc} C^\infty(M) & \longrightarrow & \chi(M) \\ H & \longmapsto & X_H \ . \end{array}$$

Definition 1.3.2. *A* **hamiltonian system** *is a triple (M, ω, H), where (M, ω) is a symplectic manifold and $H \in C^\infty(M)$ is a function, called the* **hamiltonian function***.*

Exercise 3
Show that $\{f, H\} = 0$ if and only if f is constant along integral curves of X_H.

A function f as in the previous exercise is called an **integral of motion** (or a *first integral* or a *constant of motion*) for the hamiltonian system (M, ω, H).

In general, hamiltonian systems do not admit integrals of motion which are *independent* of the hamiltonian function. Functions $f_1, \ldots, f_n$ on M are said to be **independent** if their differentials $(df_1)_p, \ldots, (df_n)_p$ are linearly independent at all points p in some open dense subset of M. Loosely speaking, a hamiltonian system is *(completely) integrable* if it has as many commuting integrals of motion as possible. Commutativity is with respect to the Poisson bracket. Notice that, if $f_1, \ldots, f_n$ are commuting integrals of motion for a hamiltonian system (M, ω, H), then, at each $p \in M$, their hamiltonian vector fields generate an **isotropic** subspace of T_pM:

$$\omega(X_{f_i}, X_{f_j}) = \{f_i, f_j\} = 0 \ .$$

If $f_1, \ldots, f_n$ are independent at p, then, by symplectic linear algebra, n can be at most half the dimension of M.

[1] The Lie bracket of two vector fields is their commutator, where they are regarded as first-order differential operators. A bilinear function $\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is a **Lie bracket** if it is antisymmetric, i.e., $\{f, g\} = -\{g, f\}$, $\forall f, g \in C^\infty(M)$, and satisfies the Jacobi identity:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \qquad \forall f, g, h \in C^\infty(M) \ .$$

Definition 1.3.3. *A hamiltonian system (M, ω, H) is* **(completely) integrable** *if it admits $n = \frac{1}{2} \dim M$ independent integrals of motion, $f_1 = H, f_2, \ldots, f_n$, which are pairwise in involution with respect to the Poisson bracket, i.e., $\{f_i, f_j\} = 0$, for all i, j.*

Example. A hamiltonian system (M, ω, H) where M is 4-dimensional is integrable if there is an integral of motion independent of H (the commutativity condition is automatically satisfied). ♢

For interesting examples of integrable systems, see [7].

Let (M, ω, H) be an integrable system of dimension $2n$ with integrals of motion $f_1 = H, f_2, \ldots, f_n$. Let $c \in \mathbb{R}^n$ be a regular value of $f := (f_1, \ldots, f_n)$. The corresponding level set, $f^{-1}(c)$, is a *lagrangian submanifold*. A submanifold Y of a $2n$-dimensional symplectic manifold (M, ω) is **lagrangian** if it is n-dimensional and if $i^*\omega = 0$ where $i : Y \hookrightarrow M$ is the inclusion map.

Exercise 4
By following the flows, show that if the hamiltonian vector fields $X_{f_1}, \ldots, X_{f_n}$ are complete on the level $f^{-1}(c)$ (i.e., if their flows are defined for all time), the connected components of $f^{-1}(c)$ are homogeneous spaces for $\mathbb{R}^n$, i.e., are of the form $\mathbb{R}^{n-k} \times \mathbb{T}^k$ for some $0 \le k \le n$, where $\mathbb{T}^k$ is a k-dimensional torus.

Any compact component of $f^{-1}(c)$ must hence be a torus. These compact components, when they exist, are called **Liouville tori**.

Theorem 1.3.4. (Arnold-Liouville [2]) *Let (M, ω, H) be an integrable system of dimension $2n$ with integrals of motion $f_1 = H, f_2, \ldots, f_n$. Let $c \in \mathbb{R}^n$ be a regular value of $f := (f_1, \ldots, f_n)$. The corresponding level $f^{-1}(c)$ is a lagrangian submanifold of M.*

(a) If the flows of $X_{f_1}, \ldots, X_{f_n}$ starting at a point $p \in f^{-1}(c)$ are complete, then the connected component of $f^{-1}(c)$ containing p is a homogeneous space for $\mathbb{R}^n$. Namely, there is an affine structure on that component with coordinates $\varphi_1, \ldots, \varphi_n$, known as **angle coordinates***, in which the flows of the vector fields $X_{f_1}, \ldots, X_{f_n}$ are linear.*

(b) There are coordinates $\psi_1, \ldots, \psi_n$, known as **action coordinates***, complementary to the angle coordinates such that the ψ_i's are integrals of motion and $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n$ form a Darboux chart.*

Therefore, the dynamics of an integrable system is extremely simple and the system has an explicit solution in action-angle coordinates. The proof of part (a) – the easy, yet interesting, part – of the Arnold-Liouville theorem is sketched above. For the proof of part (b), see [2, 17].

Geometrically, part (a) of the Arnold-Liouville theorem says that, in a neighborhood of the value c, the map $f : M \to \mathbb{R}^n$ collecting the given integrals of motion is a **lagrangian fibration**, i.e., it is locally trivial and its fibers are lagrangian submanifolds. The coordinates along the fibers are the angle coordinates.[2] Part (b) of the theorem guarantees the existence of coordinates on $\mathbb{R}^n$, the action coordinates, which (Poisson) commute among themselves and which satisfy $\{\varphi_i, \psi_j\} = \delta_{ij}$ with respect to the angle coordinates. Notice that, in general, the action coordinates are not the given integrals of motion because $\varphi_1, \ldots, \varphi_n, f_1, \ldots, f_n$ do not form a Darboux chart.

1.4 Hamiltonian Actions

Definition 1.4.1. *An* **action** *of a Lie group G on a manifold M is a group homomorphism*

$$\begin{array}{rccc} \psi: & G & \longrightarrow & \mathrm{Diff}(M) \\ & g & \longmapsto & \psi_g \ , \end{array}$$

where $\mathrm{Diff}(M)$ *is the group of diffeomorphisms of M. The* **evaluation map** *associated with an action $\psi : G \to \mathrm{Diff}(M)$ is*

$$\begin{array}{rccc} \mathrm{ev}_\psi: & M \times G & \longrightarrow & M \\ & (p,g) & \longmapsto & \psi_g(p) \ . \end{array}$$

The action ψ is **smooth** *if* ev_ψ *is a smooth map.*

We will always assume that an action is smooth.

Example. Complete vector fields[3] on a manifold M are in one-to-one correspondence with actions of $\mathbb{R}$ on M. The diffeomorphism $\psi_t : M \to M$ associated to $t \in \mathbb{R}$ is the time-t map $\exp tX$ defined by the flow of the vector field X. ◊

Let (M, ω) be a symplectic manifold, and G a Lie group with an action $\psi : G \to \mathrm{Diff}(M)$.

Definition 1.4.2. *The action ψ is a* **symplectic action** *if it is by symplectomorphisms, i.e.,*

$$\psi : G \longrightarrow \mathrm{Sympl}(M, \omega) \subset \mathrm{Diff}(M) \ ,$$

where $\mathrm{Sympl}(M, \omega)$ *is the group of symplectomorphisms of* (M, ω).

Examples.

1. On the symplectic 2-sphere $(S^2, d\theta \wedge dh)$ in cylindrical coordinates, the one-parameter group of diffeomorphisms given by rotation around the vertical axis, $\psi_t(\theta, h) = (\theta + t, h)$ $(t \in \mathbb{R})$ is a symplectic action of the group $S^1 \simeq \mathbb{R}/\langle 2\pi \rangle$, as it preserves the area form $d\theta \wedge dh$.

[2] The name *angle coordinates* is used even if the fibers are not tori.
[3] A vector field is **complete** if its integral curves through each point exist for *all* time.

2. On the symplectic 2-torus $(\mathbb{T}^2, d\theta_1 \wedge d\theta_2)$, the one-parameter groups of diffeomorphisms given by rotation around each circle, $\psi_{1,t}(\theta_1, \theta_2) = (\theta_1 + t, \theta_2)$ $(t \in \mathbb{R})$ and $\psi_{2,t}$ similarly defined, are symplectic actions of S^1.

$\diamondsuit$

Let (M, ω) be a symplectic manifold, G a Lie group with an action $\psi : G \to \mathrm{Diff}(M)$, and $\mathfrak{g}$ the Lie algebra of G with dual vector space $\mathfrak{g}^*$.

Definition 1.4.3. *The action ψ is a* **hamiltonian action** *if there exists a map*

$$\mu : M \longrightarrow \mathfrak{g}^*$$

satisfying the following two conditions:

- *For each $X \in \mathfrak{g}$, let $\mu^X : M \to \mathbb{R}$, $\mu^X(p) := \langle \mu(p), X \rangle$, be the component of μ along X, and let $X^\#$ be the vector field on M generated by the one-parameter subgroup $\{\exp tX \mid t \in \mathbb{R}\} \subseteq G$. Then*

$$d\mu^X = \imath_{X^\#}\omega$$

  i.e., the function μ^X is a hamiltonian function for the vector field $X^\#$.

- *The map μ is* equivariant *with respect to the given action ψ of G on M and the coadjoint action Ad^* of G on $\mathfrak{g}^*$:*

$$\mu \circ \psi_g = \mathrm{Ad}^*_g \circ \mu \ , \qquad \text{for all } g \in G \ .$$

The vector (M, ω, G, μ) is then called a **hamiltonian G-space** *and μ is called a* **moment map**.

Exercise 5
Check that complete symplectic vector fields on M are in one-to-one correspondence with symplectic actions of $\mathbb{R}$ on M, and that, similarly, complete hamiltonian vector fields on M are in one-to-one correspondence with hamiltonian actions of $\mathbb{R}$ on M.

Examples. Consider the previous set of two examples (as well as in the examples of Section 1.2). The first – regarding S^2 – is an example of a hamiltonian action of S^1 with moment map given by the height function, under a suitable identification of the dual of the Lie algebra of S^1 with $\mathbb{R}$. The second example – regarding T^2 – is not hamiltonian since the one-forms $d\theta_1$ and $d\theta_2$ are not exact. $\diamondsuit$

Exercise 6
Let G be any compact Lie group and H a closed subgroup of G, with $\mathfrak{g}$ and $\mathfrak{h}$ the respective Lie algebras. The projection $i^* : \mathfrak{g}^* \to \mathfrak{h}^*$ is the map dual to the inclusion $i : \mathfrak{h} \hookrightarrow \mathfrak{g}$. Suppose that (M, ω, G, ϕ) is a hamiltonian G-space. Show that the restriction of the G-action to H is hamiltonian with moment map

$$i^* \circ \phi : M \longrightarrow \mathfrak{h}^* \ .$$

Exercise 7
Suppose that a Lie group G acts in a hamiltonian way on two symplectic manifolds (M_j, ω_j), $j = 1, 2$, with moment maps $\mu_j : M_j \to \mathfrak{g}^*$. The product manifold $M_1 \times M_2$ has a natural product symplectic structure given by the sum of the pull-backs of the symplectic forms on each factor, via the two projections. Prove that the diagonal action of G on $M_1 \times M_2$ is hamiltonian with moment map $\mu : M_1 \times M_2 \to \mathfrak{g}^*$ given by

$$\mu(p_1, p_2) = \mu_1(p_1) + \mu_2(p_2) \ , \quad \text{for } p_j \in M_j \ .$$

From now on, we concentrate on actions of a torus $G = \mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$.

1.5 Hamiltonian Torus Actions

The coadjoint action is trivial on a torus (a product of circles $S^1 \times \cdots \times S^1$). Hence, if $G = \mathbb{T}^n$ is an n-dimensional torus with Lie algebra and its dual both identified with euclidean space, $\mathfrak{g} \simeq \mathbb{R}^n$ and $\mathfrak{g}^* \simeq \mathbb{R}^n$, a moment map for an action of G on (M, ω) is simply a map $\mu : M \longrightarrow \mathbb{R}^n$ satisfying:

- *For each basis vector X_i of $\mathbb{R}^n$, the function μ^{X_i} is a hamiltonian function for $X_i^\#$ and is invariant under the action of the torus.*

If $\mu : M \to \mathbb{R}^n$ is a moment map for a torus action, then clearly any of its translations $\mu + c$ ($c \in \mathbb{R}^n$) is also a moment map for that action. Reciprocally, any two moment maps for a given hamiltonian torus action differ by a constant.

Example. On $(\mathbb{C}, \omega_0 = \frac{i}{2} dz \wedge d\bar{z})$, consider the action of the circle $S^1 = \{t \in \mathbb{C} : |t| = 1\}$ by rotations

$$\psi_t(z) = t^k z \ , \qquad t \in S^1 \ ,$$

where $k \in \mathbb{Z}$ is fixed. The action $\psi : S^1 \to \mathrm{Diff}(\mathbb{C})$ is hamiltonian with moment map $\mu : \mathbb{C} \to \mathfrak{g}^* \simeq \mathbb{R}$ given by

$$\mu(z) = -\tfrac{1}{2} k |z|^2 \ .$$

This can be easily checked in polar coordinates, since $\omega_0 = r\, dr \wedge d\theta$, $\mu(re^{i\theta}) = -\frac{1}{2} k r^2$ and the vector field on $\mathbb{C}$ corresponding to the generator 1 of $\mathfrak{g} \simeq \mathbb{R}$ is $X^\# = k \frac{\partial}{\partial \theta}$. ♢

Exercise 8
Let $\mathbb{T}^n = \{(t_1, \ldots, t_n) \in \mathbb{C}^n : |t_j| = 1, \text{ for all } j\}$ be a torus acting diagonally on $\mathbb{C}^n$ by

$$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (t_1^{k_1} z_1, \ldots, t_n^{k_n} z_n) \ ,$$

where $k_1, \ldots, k_n \in \mathbb{Z}$ are fixed. Check that this action is hamiltonian with moment map $\mu : \mathbb{C}^n \to \mathfrak{g}^* \simeq \mathbb{R}^n$ given by

$$\mu(z_1, \ldots, z_n) = -\tfrac{1}{2}(k_1 |z_1|^2, \ldots, k_n |z_n|^2) \ (\ + \text{constant}\ ) \ .$$

Theorem 1.5.1. (Atiyah [3], Guillemin-Sternberg [25]) *Let (M,ω) be a compact connected symplectic manifold, and let $\mathbb{T}^m$ be an m-torus. Suppose that $\psi : \mathbb{T}^m \to \mathrm{Sympl}(M,\omega)$ is a hamiltonian action with moment map $\mu : M \to \mathbb{R}^m$. Then:*

(a) the levels of μ are connected;

(b) the image of μ is convex;

(c) the image of μ is the convex hull of the images of the fixed points of the action.

The image $\mu(M)$ of the moment map is called the **moment polytope**. A proof of Theorem 1.5.1 can be found in [36].

An action of a group G on a manifold M is called **effective** if it is injective as a map $G \to \mathrm{Diff}(M)$, i.e., each group element $g \neq e$ moves at least one point, that is, $\cap_{p\in M} G_p = \{e\}$, where $G_p = \{g \in G \mid g \cdot p = p\}$ is the stabilizer of p.

Exercise 9
Suppose that $\mathbb{T}^m$ acts linearly on $(\mathbb{C}^n, \omega_0)$. Let $\lambda^{(1)}, \ldots, \lambda^{(n)} \in \mathbb{Z}^m$ be the weights appearing in the corresponding weight space decomposition (further discussed in Section 6.2), that is,

$$\mathbb{C}^n \simeq \bigoplus_{k=1}^{n} V_{\lambda^{(k)}} \ ,$$

where, for $\lambda^{(k)} = (\lambda_1^{(k)}, \ldots, \lambda_m^{(k)})$, $\mathbb{T}^m$ acts on the complex line $V_{\lambda^{(k)}}$ by

$$(e^{it_1}, \ldots, e^{it_m}) \cdot v = e^{i\sum_j \lambda_j^{(k)} t_j} v \ , \qquad \forall v \in V_{\lambda^{(k)}} \ , \ \forall k = 1, \ldots, n \ .$$

(a) Show that, if the action is effective, then $m \leq n$ and the weights $\lambda^{(1)}, \ldots, \lambda^{(n)}$ are part of a $\mathbb{Z}$-basis of $\mathbb{Z}^m$.

(b) Show that, if the action is symplectic (hence, hamiltonian), then the weight spaces $V_{\lambda^{(k)}}$ are symplectic subspaces.

(c) Show that, if the action is hamiltonian, then a moment map is given by

$$\mu(v) = -\tfrac{1}{2} \sum_{k=1}^{n} \lambda^{(k)} ||v_{\lambda^{(k)}}||^2 \ (+ \text{constant}) \ ,$$

where $|| \cdot ||$ is the standard norm[a] and $v = v_{\lambda^{(1)}} + \ldots + v_{\lambda^{(n)}}$ is the weight space decomposition. Cf. Exercise 8.

(d) Conclude that, if $\mathbb{T}^n$ acts on $\mathbb{C}^n$ in a linear, effective and hamiltonian way, then any moment map μ is a submersion, i.e., each differential $d\mu_v : \mathbb{C}^n \to \mathbb{R}^n$ $(v \in \mathbb{C}^n)$ is surjective.

[a] Notice that the standard inner product satisfies $(v,w) = \omega_0(v, Jv)$ where $J\frac{\partial}{\partial z} = i\frac{\partial}{\partial z}$ and $J\frac{\partial}{\partial \bar{z}} = -i\frac{\partial}{\partial \bar{z}}$. In particular, the standard norm is invariant for a symplectic complex-linear action.

The following two results use the crucial fact that any effective action $\mathbb{T}^m \to \mathrm{Diff}(M)$ has orbits of dimension m; a proof may be found in [10].

Corollary 1.5.2. *Under the conditions of the convexity theorem, if the $\mathbb{T}^m$-action is effective, then there must be at least $m+1$ fixed points.*

Proof. At a point p of an m-dimensional orbit the moment map is a submersion, i.e., $(d\mu_1)_p, \ldots, (d\mu_m)_p$ are linearly independent. Hence, $\mu(p)$ is an interior point of $\mu(M)$, and $\mu(M)$ is a nondegenerate convex polytope. Any nondegenerate convex polytope in $\mathbb{R}^m$ must have at least $m+1$ vertices. The vertices of $\mu(M)$ are images of fixed points. □

Theorem 1.5.3. *Let $(M, \omega, \mathbb{T}^m, \mu)$ be a hamiltonian $\mathbb{T}^m$-space. If the $\mathbb{T}^m$-action is effective, then* $\dim M \geq 2m$.

Proof. Since the moment map is constant on an orbit $\mathcal{O}$, for $p \in \mathcal{O}$ the exterior derivative

$$d\mu_p : T_pM \longrightarrow \mathfrak{g}^*$$

maps $T_p\mathcal{O}$ to 0. Thus

$$T_p\mathcal{O} \subseteq \ker d\mu_p = (T_p\mathcal{O})^\omega \ ,$$

where $(T_p\mathcal{O})^\omega$ is the symplectic orthogonal of $T_p\mathcal{O}$. This shows that orbits $\mathcal{O}$ of a hamiltonian torus action are always isotropic submanifolds of M. In particular, by symplectic linear algebra we have that $\dim \mathcal{O} \leq \frac{1}{2} \dim M$. Now consider an m-dimensional orbit. □

1.6 Symplectic Toric Manifolds

Definition 1.6.1. *A* **symplectic toric manifold** *is a compact connected symplectic manifold (M, ω) equipped with an effective hamiltonian action of a torus $\mathbb{T}$ of dimension equal to half the dimension of the manifold,*

$$\dim \mathbb{T} = \frac{1}{2} \dim M \ ,$$

and with a choice of a corresponding moment map μ.

Exercise 10
Show that an effective hamiltonian action of a torus $\mathbb{T}^n$ on a $2n$-dimensional symplectic manifold gives rise to an integrable system.

Hint: The coordinates of the moment map are commuting integrals of motion.

Definition 1.6.2. *Two symplectic toric manifolds, $(M_i, \omega_i, \mathbb{T}_i, \mu_i)$, $i = 1, 2$, are* **equivalent** *if there exists an isomorphism $\lambda : \mathbb{T}_1 \to \mathbb{T}_2$ and a λ-equivariant symplectomorphism $\varphi : M_1 \to M_2$ such that $\mu_1 = \mu_2 \circ \varphi$.*

Equivalent symplectic toric manifolds are often undistinguished.

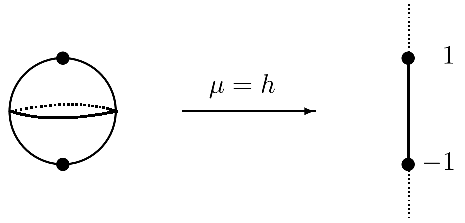
Examples of symplectic toric manifolds.

1. The circle S^1 acts on the 2-sphere $(S^2, \omega_{\text{standard}} = d\theta \wedge dh)$ by rotations

$$e^{it} \cdot (\theta, h) = (\theta + t, h)$$

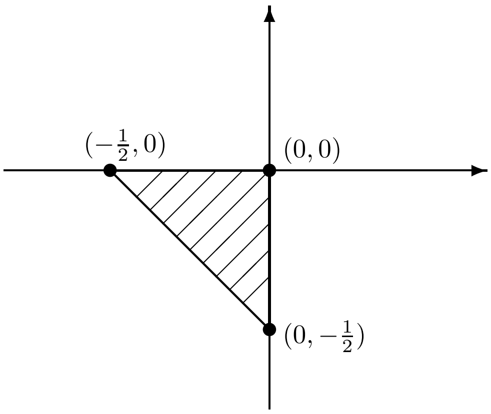
with moment map $\mu = h$ equal to the height function and moment polytope $[-1, 1]$.

Equivalently, the circle S^1 acts on $\mathbb{P}^1 = \mathbb{C}^2 - 0/\sim$ with the Fubini-Study form $\omega_{\text{FS}} = \frac{1}{4}\omega_{\text{standard}}$, by $e^{it} \cdot [z_0 : z_1] = [z_0 : e^{it} z_1]$. This is hamiltonian with moment map $\mu[z_0 : z_1] = -\frac{1}{2} \cdot \frac{|z_1|^2}{|z_0|^2+|z_1|^2}$, and moment polytope $\left[-\frac{1}{2}, 0\right]$.



2. Let $(\mathbb{P}^2, \omega_{\text{FS}})$ be 2-(complex-)dimensional complex projective space equipped with the Fubini-Study form defined in Section 2.3. The $\mathbb{T}^2$-action on $\mathbb{P}^2$ by $(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0 : z_1 : z_2] = [z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2]$ has moment map

$$\mu[z_0 : z_1 : z_2] = -\frac{1}{2} \left(\frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2} \right) .$$



The fixed points get mapped as

$$\begin{array}{lcl} [1:0:0] & \longmapsto & (0,0) \\ [0:1:0] & \longmapsto & \left(-\frac{1}{2}, 0\right) \\ [0:0:1] & \longmapsto & \left(0, -\frac{1}{2}\right) \end{array}$$

Notice that the stabilizer of a preimage of the edges is S^1, while the action is free at preimages of interior points of the moment polytope.

Exercise 11
Compute a moment polytope for the $\mathbb{T}^3$-action on $\mathbb{P}^3$ as

$$(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2 : e^{i\theta_3} z_3] \ .$$

Exercise 12
Compute a moment polytope for the $\mathbb{T}^2$-action on $\mathbb{P}^1 \times \mathbb{P}^1$ as

$$(e^{i\theta}, e^{i\eta}) \cdot ([z_0 : z_1], [w_0 : w_1]) = ([z_0 : e^{i\theta} z_1], [w_0 : e^{i\eta} w_1]) \ .$$

Chapter 2

Classification

Recall that a $2n$-dimensional symplectic toric manifold is a compact connected symplectic manifold (M^{2n}, ω) equipped with an effective hamiltonian action of an n-torus $\mathbb{T}^n$ and with a corresponding moment map $\mu : M \to \mathbb{R}^n$. In this lecture we describe the classification of equivalence classes of symplectic toric manifolds by their moment polytopes $\mu(M)$. Symplectic reduction is the quotienting technique which we use for the construction of a symplectic toric manifold out of an appropriate polytope, thus proving the existence part in the classification theorem.

2.1 Delzant's Theorem

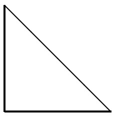
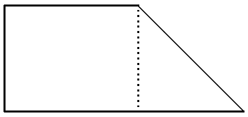
We now define the class of polytopes[1] which arise in the classification of symplectic toric manifolds.

Definition 2.1.1. *A* **Delzant polytope** Δ *in* $\mathbb{R}^n$ *is a polytope satisfying:*

- **simplicity***, i.e., there are* n *edges meeting at each vertex;*
- **rationality***, i.e., the edges meeting at the vertex* p *are rational in the sense that each edge is of the form* $p + tu_i$, $t \geq 0$, *where* $u_i \in \mathbb{Z}^n$*;*
- **smoothness***, i.e., for each vertex, the corresponding* $u_1, \ldots, u_n$ *can be chosen to be a* $\mathbb{Z}$*-basis of* $\mathbb{Z}^n$.

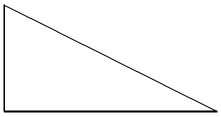
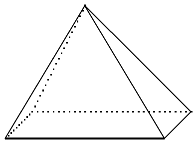
[1] A **polytope** in $\mathbb{R}^n$ is the convex hull of a finite number of points in $\mathbb{R}^n$. A **convex polyhedron** is a subset of $\mathbb{R}^n$ which is the intersection of a finite number of affine half-spaces. Hence, polytopes coincide with bounded convex polyhedra.

Examples of Delzant polytopes in $\mathbb{R}^2$:

The dotted vertical line in the trapezoidal example is there just to stress that it is a picture of a rectangle plus an isosceles triangle. For "taller" triangles, smoothness would be violated. "Wider" triangles (with integral slope) may still be Delzant. The family of the Delzant trapezoids of this type, starting with the rectangle, correspond, under the Delzant construction, to *Hirzebruch surfaces*; see Lecture 3. ♢

Examples of polytopes which are not Delzant:

The picture on the left fails the smoothness condition, since the triangle is not isosceles, whereas the one on the right fails the simplicity condition. ♢

Delzant's theorem classifies (equivalence classes of) symplectic toric manifolds in terms of the combinatorial data encoded by a Delzant polytope.

Theorem 2.1.2. (Delzant [15]) *Toric manifolds are classified by Delzant polytopes. More specifically, the bijective correspondence between these two sets is given by the moment map:*

$$\begin{array}{ccc} \{\textit{toric manifolds}\} & \stackrel{1-1}{\longrightarrow} & \{\textit{Delzant polytopes}\} \\ (M^{2n}, \omega, \mathbb{T}^n, \mu) & \longmapsto & \mu(M)\ . \end{array}$$

In Section 2.5, we describe the construction which proves the (easier) existence part, or surjectivity, in Delzant's theorem. In order to prepare that, we will next give an algebraic description of Delzant polytopes.

Let Δ be a Delzant polytope in $(\mathbb{R}^n)^*$[2] and with d facets.[3] Let $v_i \in \mathbb{Z}^n$,

[2] Although we identify $\mathbb{R}^n$ with its dual via the euclidean inner product, it may be more clear to see Δ in $(\mathbb{R}^n)^*$ for Delzant's construction.

[3] A **face** of a polytope Δ is a set of the form $F = P \cap \{x \in \mathbb{R}^n \mid f(x) = c\}$ where $c \in \mathbb{R}$

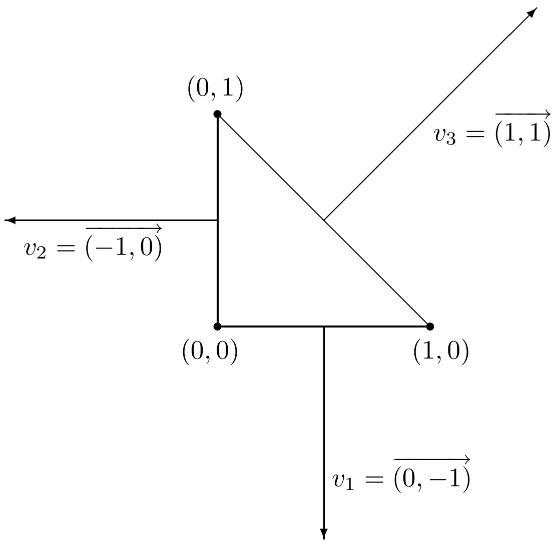
$i = 1, \ldots, d$, be the primitive[4] outward-pointing normal vectors to the facets of Δ. Then we can describe Δ as an intersection of halfspaces

$$\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, v_i \rangle \leq \lambda_i,\ i = 1, \ldots, d\} \quad \text{for some } \lambda_i \in \mathbb{R}\ .$$

Example. For the picture below, we have

$$\begin{array}{rcl} \Delta & = & \{x \in (\mathbb{R}^2)^* \mid x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq 1\} \\ & = & \{x \in (\mathbb{R}^2)^* \mid \langle x, (-1,0)\rangle \leq 0\ ,\ \langle x, (0,-1)\rangle \leq 0\ ,\ \langle x, (1,1)\rangle \leq 1\}\ . \end{array}$$

$\diamondsuit$



2.2 Orbit Spaces

Let $\psi : G \to \mathrm{Diff}(M)$ be any action.

Definition 2.2.1. *The* **orbit** *of G through $p \in M$ is $\{\psi_g(p) \mid g \in G\}$. The* **stabilizer** *(or* isotropy*) of $p \in M$ is $G_p := \{g \in G \mid \psi_g(p) = p\}$.*

---

and $f \in (\mathbb{R}^n)^*$ satisfies $f(x) \geq c$, $\forall x \in P$. A **facet** of an n-dimensional polytope is an $(n-1)$-dimensional face.

[4] A lattice vector $v \in \mathbb{Z}^n$ is **primitive** if it cannot be written as $v = ku$ with $u \in \mathbb{Z}^n$, $k \in \mathbb{Z}$ and $|k| > 1$; for instance, $(1,1)$, $(4,3)$, $(1,0)$ are primitive, but $(2,2)$, $(4,6)$ are not.

Exercise 13
If q is in the orbit of p, then G_q and G_p are conjugate subgroups.

Definition 2.2.2. *We say that the action of G on M is:*

- **transitive** *if there is just one orbit,*
- **free** *if all stabilizers are trivial $\{e\}$,*
- **locally free** *if all stabilizers are discrete.*

Let $\sim$ be the orbit equivalence relation; for $p, q \in M$,

$$p \sim q \quad \Longleftrightarrow \quad p \text{ and } q \text{ are on the same orbit.}$$

The space of orbits $M/G := M/\sim$ is called the **orbit space**. Let

$$\begin{array}{rccl} \pi : & M & \longrightarrow & M/G \\ & p & \longmapsto & \text{orbit through } p \end{array}$$

be the **point-orbit projection**.

We equip M/G with the weakest topology for which π is continuous, i.e., $\mathcal{U} \subseteq M/G$ is open if and only if $\pi^{-1}(\mathcal{U})$ is open in M. This is called the **quotient topology**. This topology can be "bad." For instance:

Example. Let $G = \mathbb{C}\backslash\{0\}$ act on $M = \mathbb{C}^n$ by

$$\lambda \longmapsto \psi_\lambda = \text{multiplication by } \lambda \ .$$

The orbits are the punctured complex lines (through non-zero vectors $z \in \mathbb{C}^n$), plus one "unstable" orbit through 0, which has a single point. The orbit space is

$$M/G = \mathbb{P}^{n-1} \sqcup \{\text{point}\} \ .$$

The quotient topology restricts to the usual topology on $\mathbb{P}^{n-1}$. The only open set containing $\{\text{point}\}$ in the quotient topology is the full space, hence the topology in M/G is *not* Hausdorff.

However, it suffices to remove 0 from $\mathbb{C}^n$ to obtain a Hausdorff orbit space: $\mathbb{P}^{n-1}$. Then there is also a compact (yet not complex) description of the orbit space by taking only unit vectors under the action of the circle subgroup:

$$\mathbb{P}^{n-1} = \Big(\mathbb{C}^n\backslash\{0\}\Big)\Big/\Big(\mathbb{C}\backslash\{0\}\Big) = S^{2n-1}/S^1 \ .$$

$\diamondsuit$

2.3 Symplectic Reduction

Let $\omega = \frac{i}{2}\sum dz_k \wedge d\bar{z}_k = \sum dx_k \wedge dy_k = \sum r_k dr_k \wedge d\theta_k$ be the standard symplectic form on $\mathbb{C}^n$. Consider the following S^1-action on $(\mathbb{C}^n, \omega)$:

$$t \in S^1 \longmapsto \psi_t = \text{multiplication by } t \ .$$

The action ψ is hamiltonian with moment map

$$\begin{array}{rccl} \mu : & \mathbb{C}^n & \longrightarrow & \mathbb{R} \\ & z & \longmapsto & -\frac{||z||^2}{2} + \text{constant} \end{array}$$

since

$$\begin{aligned} d\mu &= -\tfrac{1}{2} d(\textstyle\sum r_k^2) \ , \\ X^\# &= \frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2} + \cdots + \frac{\partial}{\partial\theta_n} \quad \text{and} \\ \imath_{X^\#}\omega &= -\textstyle\sum r_k dr_k = -\frac{1}{2}\sum dr_k^2 \ . \end{aligned}$$

If we choose the constant to be $\frac{1}{2}$, then $\mu^{-1}(0) = S^{2n-1}$ is the unit sphere. The orbit space of the zero level of the moment map is

$$\mu^{-1}(0)/S^1 = S^{2n-1}/S^1 = \mathbb{P}^{n-1} \ .$$

Moreover, this construction induces a symplectic form on $\mathbb{P}^{n-1}$, as a particular instance of the following major theorem.

Theorem 2.3.1. (Marsden-Weinstein [34], Meyer [37]) *Let (M, ω, G, μ) be a hamiltonian G-space for a compact Lie group G. Let $i : \mu^{-1}(0) \hookrightarrow M$ be the inclusion map. Assume that G acts freely on $\mu^{-1}(0)$. Then*

(a) the orbit space $M_{\text{red}} = \mu^{-1}(0)/G$ is a manifold,

(b) $\pi : \mu^{-1}(0) \to M_{\text{red}}$ is a principal G-bundle, and

(c) there is a symplectic form ω_{red} on M_{red} satisfying $i^*\omega = \pi^*\omega_{\text{red}}$.

For a proof of Theorem 2.3.1, see for instance [12].

Definition 2.3.2. *The pair $(M_{\text{red}}, \omega_{\text{red}})$ is called the* **symplectic reduction** *of (M, ω) with respect to G and μ (or the* reduced space*, or the* symplectic quotient*, or the* Marsden-Weinstein-Meyer quotient*, etc.).*

Remark. When M is Kähler, i.e., has a compatible complex structure, and the action of G preserves the complex structure, then the symplectic reduction has a natural Kähler structure. ♢

Example. Consider the S^1-action on $(\mathbb{R}^{2n+2}, \omega_0)$ which, under the usual identification of $\mathbb{R}^{2n+2}$ with $\mathbb{C}^{n+1}$, corresponds to multiplication by e^{it}. This action is hamiltonian with a moment map $\mu : \mathbb{C}^{n+1} \to \mathbb{R}$ given by

$$\mu(z) = -\tfrac{1}{2}||z||^2 + \tfrac{1}{2} \ .$$

Symplectic reduction yields complex projective space $\mu^{-1}(0)/S^1 = \mathbb{P}^n$ equipped with the so-called **Fubini-Study symplectic form** $\omega_{\text{red}} = \omega_{\text{FS}}$. $\diamondsuit$

Exercise 14
Recall that $\mathbb{P}^1 \simeq S^2$ as real 2-dimensional manifolds. Check that

$$\omega_{\text{FS}} = \frac{1}{4}\omega_{\text{standard}} \ ,$$

where $\omega_{\text{standard}} = d\theta \wedge dh$ is the standard area form on S^2.

2.4 Extensions of Symplectic Reduction

We consider three basic extensions of the procedure of symplectic reduction.

1. **Reduction for product groups.**

   Let G_1 and G_2 be compact connected Lie groups and let $G = G_1 \times G_2$. Then $\mathfrak{g} \simeq \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $\mathfrak{g}^* \simeq \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*$. Suppose that (M, ω, G, ν) is a hamiltonian G-space with moment map

   $$\nu : M \longrightarrow \mathfrak{g}_1^* \oplus \mathfrak{g}_2^* \ .$$

   Write $\nu = (\nu_1, \nu_2)$ where $\nu_i : M \to \mathfrak{g}_i^*$ for $i = 1, 2$. The fact that ν is equivariant implies that ν_1 is invariant under G_2 and ν_2 is invariant under G_1. Now reduce (M, ω) with respect to the G_1-action. Let

   $$Z_1 = \nu_1^{-1}(0) \ .$$

   Assume that G_1 acts freely on Z_1. Let $M_1 = Z_1/G_1$ be the reduced space and let ω_1 be the corresponding reduced symplectic form. The action of G_2 on Z_1 commutes with the G_1-action. Since G_2 preserves ω, it follows that G_2 acts symplectically on (M_1, ω_1). Since G_1 preserves ν_2, G_1 also preserves $\nu_2 \circ \iota_1 : Z_1 \to \mathfrak{g}_2^*$, where $\iota_1 : Z_1 \hookrightarrow M$ is inclusion. Thus $\nu_2 \circ \iota_1$ is constant on fibers of $Z_1 \stackrel{p_1}{\to} M_1$. We conclude that there exists a smooth map $\mu_2 : M_1 \to \mathfrak{g}_2^*$ such that $\mu_2 \circ p_1 = \nu_2 \circ \iota_1$.

Exercise 15
Show that:

(a) the map μ_2 is a moment map for the action of G_2 on (M_1, ω_1), and

(b) if G acts freely on $\nu^{-1}(0,0)$, then G_2 acts freely on $\mu_2^{-1}(0)$, and there is a natural symplectomorphism

$$\nu^{-1}(0,0)/G \simeq \mu_2^{-1}(0)/G_2 \ .$$

This technique of performing reduction with respect to one factor of a product group at a time is called **reduction in stages**. It may be extended to reduction by a normal subgroup $H \subset G$ and by the corresponding quotient group G/H.

2. **Reduction at other levels.**

Suppose that a compact Lie group G acts on a symplectic manifold (M, ω) in a hamiltonian way with moment map $\mu : M \to \mathfrak{g}^*$. Let $\xi \in \mathfrak{g}^*$. To reduce at the level ξ of μ, we need $\mu^{-1}(\xi)$ to be preserved by G, or else take the G-orbit of $\mu^{-1}(\xi)$, or equivalently take the inverse image $\mu^{-1}(\mathcal{O}_\xi)$ of the coadjoint orbit through ξ, or else take the quotient by the maximal subgroup of G which preserves $\mu^{-1}(\xi)$. Of course the level 0 is always preserved. Also, when G is a torus, any level is preserved and reduction at ξ for the moment map μ, is equivalent to reduction at 0 for a shifted moment map $\phi : M \to \mathfrak{g}^*$, $\phi(p) := \mu(p) - \xi$.

3. **Orbifold singularities.**

Roughly speaking, orbifolds (introduced by Satake in [42]) are singular manifolds where each singularity is locally modeled on $\mathbb{R}^m/\Gamma$, for some finite group $\Gamma \subset \mathrm{GL}(m;\mathbb{R})$. For the precise definition, let $|M|$ be a Hausdorff topological space satisfying the second axiom of countability.

Definition 2.4.1. *An* **orbifold chart** *on $|M|$ is a triple $(\mathcal{V}, \Gamma, \varphi)$, where $\mathcal{V}$ is a connected open subset of some euclidean space $\mathbb{R}^m$, Γ is a finite group which acts linearly on $\mathcal{V}$ so that the set of points where the action is not free has codimension at least two, and $\varphi : \mathcal{V} \to |M|$ is a Γ-invariant map inducing a homeomorphism from $\mathcal{V}/\Gamma$ onto its image $\mathcal{U} \subset |M|$. An* **orbifold atlas** *$\mathcal{A}$ for $|M|$ is a collection of orbifold charts on $|M|$ such that: the collection of images $\mathcal{U}$ forms a basis of open sets in $|M|$, and the charts are compatible in the sense that, whenever two charts $(\mathcal{V}_1, \Gamma_1, \varphi_1)$ and $(\mathcal{V}_2, \Gamma_2, \varphi_2)$ satisfy $\mathcal{U}_1 \subseteq \mathcal{U}_2$, there exists an injective homomorphism $\lambda : \Gamma_1 \to \Gamma_2$ and a λ-equivariant open embedding $\psi : \mathcal{V}_1 \to \mathcal{V}_2$ such that $\varphi_2 \circ \psi = \varphi_1$. Two orbifold atlases are* **equivalent** *if their union is still an atlas. An m-dimensional* **orbifold** *M is a Hausdorff topological space $|M|$ satisfying the second axiom of countability, plus an equivalence class of orbifold atlases on $|M|$.*

Notice that we do not require the action of each group Γ to be effective. Given a point p on an orbifold M, let $(\mathcal{V}, \Gamma, \varphi)$ be an orbifold chart for a neighborhood $\mathcal{U}$ of p. The **orbifold structure group** of p, Γ_p, is (the isomorphism class of) the isotropy group of a pre-image of p under ϕ. We may always choose an orbifold chart $(\mathcal{V}, \Gamma, \varphi)$ such that $\varphi^{-1}(p)$ is a single point (which is fixed by Γ). In this case $\Gamma \simeq \Gamma_p$, and we say that $(\mathcal{V}, \Gamma, \varphi)$ is a **structure chart** for p.

An ordinary manifold is a special case of orbifold where each group Γ is the identity group. Quotients of manifolds by locally free actions of Lie groups are orbifolds. In fact, any orbifold M has a presentation of this form obtained as follows. Given a structure chart $(\mathcal{V}, \Gamma, \varphi)$ for $p \in M$ with image $\mathcal{U}$, the **orbifold tangent space** at p is the quotient of the tangent space to $\mathcal{V}$ at $\varphi^{-1}(p)$ by the induced action of Γ:

$$T_pM := T_{\varphi^{-1}(p)}\mathcal{V}/\Gamma \ .$$

The collection of the orbifold tangent spaces at all p, builds up the **orbifold tangent bundle** TM, which has a natural structure of smooth manifold outside the zero section. The general linear group $\mathrm{GL}(m;\mathbb{R})$ acts locally freely on $TM \setminus 0$, and $M \simeq (TM-0)/\mathrm{GL}(m;\mathbb{R})$. Choosing a riemannian metric and taking the orthonormal frame bundle, $O(TM)$, we present M as $O(TM)/\mathrm{O}(m)$.

Example. Let $G = \mathbb{T}^n$ be an n-torus acting on a symplectic manifold (M,ω) in a hamiltonian way with moment map $\mu : M \to \mathfrak{g}^*$. For any $\xi \in \mathfrak{g}^*$, the level $\mu^{-1}(\xi)$ is preserved by the $\mathbb{T}^n$-action. Suppose that ξ is a regular value of μ.[5] Then $\mu^{-1}(\xi)$ is a submanifold of codimension n. Let G_p be the stabilizer of p, and $\mathfrak{g}_p$ its Lie algebra. Note that

$$\begin{array}{rcl} \xi \text{ regular} & \Longleftrightarrow & d\mu_p \text{ is surjective at all } p \in \mu^{-1}(\xi) \\ & \Longleftrightarrow & \mathfrak{g}_p = 0 \ \text{ for all } p \in \mu^{-1}(\xi) \\ & \Longleftrightarrow & \text{the stabilizers on } \mu^{-1}(\xi) \text{ are finite} \ . \end{array}$$

By the slice theorem (see, for instance, [6, 12]), near $\mathcal{O}_p$ the orbit space $\mu^{-1}(\xi)/G$ is modeled by S/G_p, where S is a G_p-invariant disk in $\mu^{-1}(\xi)$ through p and transverse to $\mathcal{O}_p$. Hence, $\mu^{-1}(\xi)/G$ is an orbifold. ♢

Example. Consider the S^1-action on $\mathbb{C}^2$ by $e^{i\theta} \cdot (z_1, z_2) = (e^{ik\theta}z_1, e^{i\theta}z_2)$ for some fixed integer $k \geq 2$. This is hamiltonian with moment map

$$\begin{array}{rccl} \mu : & \mathbb{C}^2 & \longrightarrow & \mathbb{R} \\ & (z_1,z_2) & \longmapsto & -\frac{1}{2}(k|z_1|^2 + |z_2|^2) \ . \end{array}$$

Any $\xi < 0$ is a regular value and $\mu^{-1}(\xi)$ is a 3-dimensional ellipsoid. The stabilizer of $(z_1, z_2) \in \mu^{-1}(\xi)$ is $\{1\}$ if $z_2 \neq 0$, and is

$$\mathbb{Z}_k = \left\{ e^{i\frac{2\pi\ell}{k}} \mid \ell = 0, 1, \ldots, k-1 \right\}$$

[5]By Sard's theorem, the singular values of μ form a set of measure zero.

if $z_2 = 0$. The reduced space $\mu^{-1}(\xi)/S^1$ is called a **teardrop** orbifold or *conehead*; it has one **cone** (also known as a *dunce cap*) singularity with cone angle $\frac{2\pi}{k}$, that is, a point with orbifold structure group $\mathbb{Z}_k$. $\diamondsuit$

Example. Let S^1 act on $\mathbb{C}^2$ by $e^{i\theta} \cdot (z_1, z_2) = (e^{ik\theta} z_1, e^{i\ell\theta} z_2)$ for some integers $k, \ell \geq 2$. Suppose that k and ℓ are relatively prime. Then

$$\begin{array}{rll} (z_1, 0) & \text{has stabilizer } \mathbb{Z}_k & (\text{for } z_1 \neq 0) \ , \\ (0, z_2) & \text{has stabilizer } \mathbb{Z}_\ell & (\text{for } z_2 \neq 0) \ , \\ (z_1, z_2) & \text{has stabilizer } \{1\} & (\text{for } z_1, z_2 \neq 0) \ . \end{array}$$

The quotient $\mu^{-1}(\xi)/S^1$ is called a **football** orbifold. It has two cone singularities, one with angle $\frac{2\pi}{k}$ and another with angle $\frac{2\pi}{\ell}$. $\diamondsuit$

Example. More generally, the reduced spaces of S^1 acting on $\mathbb{C}^n$ by

$$e^{i\theta} \cdot (z_1, \dots, z_n) = (e^{ik_1\theta} z_1, \dots, e^{ik_n\theta} z_n) \ ,$$

are called **weighted** (or *twisted*) **projective spaces**. $\diamondsuit$

The differential-geometric notions of vector fields, differential forms, exterior differentiation, group actions, etc., extend naturally to orbifolds by gluing corresponding local Γ-invariant or Γ-equivariant objects. In particular, a **symplectic orbifold** is a pair (M, ω) where M is an orbifold and ω is a closed 2-form on M which is nondegenerate at every point of M.

Definition 2.4.2. *A* **symplectic toric orbifold** *is a compact connected symplectic orbifold (M, ω) equipped with an effective hamiltonian action of a torus $\mathbb{T}$ of dimension equal to half the dimension of the orbifold,*

$$\dim \mathbb{T} = \frac{1}{2} \dim M \ ,$$

and with a choice of a corresponding moment map μ.

Symplectic toric orbifolds have been classified by Lerman and Tolman [33] in a theorem which generalizes Delzant's theorem: a symplectic toric orbifold is determined by its moment polytope plus a positive integer label attached to each of the polytope facets. The polytopes which occur in the Lerman-Tolman classification are more general than the Delzant polytopes in the sense that only simplicity and rationality are required; the edge vectors $u_1, \dots, u_n$ need only form a rational basis of $\mathbb{Z}^n$. In the case where the integer labels are all equal to 1, the failure of the polytope smoothness accounts for all orbifold singularities. Throughout the rest of these notes, we concentrate on the manifold case.

2.5 Delzant's Construction

Following [15, 24], we prove the existence part (or surjectivity) in Delzant's theorem, by using symplectic reduction to associate to an n-dimensional Delzant polytope Δ a symplectic toric manifold $(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu_\Delta)$.

Let Δ be a Delzant polytope with d facets. Let $v_i \in \mathbb{Z}^n$, $i = 1, \ldots, d$, be the primitive outward-pointing normal vectors to the facets. For some $\lambda_i \in \mathbb{R}$, we can write

$$\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, v_i \rangle \leq \lambda_i,\ i = 1, \ldots, d\} \ .$$

Let $e_1 = (1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, 1)$ be the standard basis of $\mathbb{R}^d$. Consider

$$\begin{array}{rccc} \pi : & \mathbb{R}^d & \longrightarrow & \mathbb{R}^n \\ & e_i & \longmapsto & v_i \ . \end{array}$$

Lemma 2.5.1. *The map π is onto and maps $\mathbb{Z}^d$ onto $\mathbb{Z}^n$.*

Proof. The set $\{e_1, \ldots, e_d\}$ is a basis of $\mathbb{Z}^d$. The set $\{v_1, \ldots, v_d\}$ spans $\mathbb{Z}^n$ for the following reason. At a vertex p, the edge vectors $u_1, \ldots, u_n \in (\mathbb{R}^n)^*$, form a basis for $(\mathbb{Z}^n)^*$ which, by a change of basis if necessary, we may assume is the standard basis. Then the corresponding primitive normal vectors to the facets meeting at p are symmetric (in the sense of multiplication by -1) to the u_i's, hence form a basis of $\mathbb{Z}^n$. □

Therefore, π induces a surjective map, still called π, between tori:

$$\begin{array}{ccccc} \mathbb{R}^d/(2\pi\mathbb{Z}^d) & \stackrel{\pi}{\longrightarrow} & \mathbb{R}^n/(2\pi\mathbb{Z}^n) & & \\ \| & & \| & & \\ \mathbb{T}^d & \longrightarrow & \mathbb{T}^n & \longrightarrow & 1 \ . \end{array}$$

The kernel N of π is a $(d-n)$-dimensional Lie subgroup of $\mathbb{T}^d$ with inclusion $i : N \hookrightarrow \mathbb{T}^d$. Let $\mathfrak{n}$ be the Lie algebra of N. The exact sequence of tori

$$1 \longrightarrow N \stackrel{i}{\longrightarrow} \mathbb{T}^d \stackrel{\pi}{\longrightarrow} \mathbb{T}^n \longrightarrow 1$$

induces an exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{n} \stackrel{i}{\longrightarrow} \mathbb{R}^d \stackrel{\pi}{\longrightarrow} \mathbb{R}^n \longrightarrow 0$$

with dual exact sequence

$$0 \longrightarrow (\mathbb{R}^n)^* \stackrel{\pi^*}{\longrightarrow} (\mathbb{R}^d)^* \stackrel{i^*}{\longrightarrow} \mathfrak{n}^* \longrightarrow 0 \ .$$

Now consider $\mathbb{C}^d$ with symplectic form $\omega_0 = \frac{i}{2} \sum dz_k \wedge d\bar{z}_k$, and standard hamiltonian action of $\mathbb{T}^d$ given by

$$(e^{it_1}, \ldots, e^{it_d}) \cdot (z_1, \ldots, z_d) = (e^{it_1} z_1, \ldots, e^{it_d} z_d) \ .$$

The moment map is $\phi : \mathbb{C}^d \longrightarrow (\mathbb{R}^d)^*$ defined by

$$\phi(z_1, \ldots, z_d) = -\frac{1}{2}(|z_1|^2, \ldots, |z_d|^2) + \text{ constant },$$

where we choose the constant to be $(\lambda_1, \ldots, \lambda_d)$. By Exercise 6, the subtorus N acts on $\mathbb{C}^d$ in a hamiltonian way with moment map

$$i^* \circ \phi : \mathbb{C}^d \longrightarrow \mathfrak{n}^* .$$

Let $Z = (i^* \circ \phi)^{-1}(0)$ be the zero-level set.

Claim 1. The set Z is compact and N acts freely on Z.

We postpone the proof of this claim until further down.

Since i^* is surjective, $0 \in \mathfrak{n}^*$ is a regular value of $i^* \circ \phi$. Hence, Z is a compact submanifold of $\mathbb{C}^d$ of (real) dimension $2d - (d - n) = d + n$. The orbit space $M_\Delta = Z/N$ is a compact manifold of (real) dimension $\dim Z - \dim N = (d + n) - (d - n) = 2n$. The point-orbit map $p : Z \to M_\Delta$ is a principal N-bundle over M_Δ. Consider the diagram

$$\begin{array}{ccc} Z & \overset{j}{\hookrightarrow} & \mathbb{C}^d \\ {\scriptstyle p}\downarrow & & \\ M_\Delta & & \end{array}$$

where $j : Z \hookrightarrow \mathbb{C}^d$ is inclusion. The Marsden-Weinstein-Meyer theorem guarantees the existence of a symplectic form ω_Δ on M_Δ satisfying

$$p^* \omega_\Delta = j^* \omega_0 .$$

Since Z is connected, the compact symplectic $2n$-dimensional manifold $(M_\Delta, \omega_\Delta)$ is also connected.

Proof of Claim 1. The set Z is clearly closed, hence in order to show that it is compact it suffices (by the Heine-Borel theorem) to show that Z is bounded. Let Δ' be the image of Δ by π^*. We will show that $\phi(Z) = \Delta'$.

Lemma 2.5.2. *Let $y \in (\mathbb{R}^d)^*$. Then:*

$$y \in \Delta' \iff y \text{ is in the image of } Z \text{ by } \phi .$$

Proof. The value y is in the image of Z by ϕ if and only if both of the following conditions hold:

1. y is in the image of ϕ;
2. $i^* y = 0$.

Using the expression for ϕ and the third exact sequence, we see that these conditions are equivalent to:

1. $\langle y, e_i \rangle \leq \lambda_i$ for $i = 1, \ldots, d$;
2. $y = \pi^*(x)$ for some $x \in (\mathbb{R}^n)^*$.

Suppose that the second condition holds, so that $y = \pi^*(x)$. Then

$$\begin{aligned}
\langle y, e_i \rangle \leq \lambda_i, \forall i \quad &\Longleftrightarrow \quad \langle \pi^*(x), e_i \rangle \leq \lambda_i, \forall i \\
&\Longleftrightarrow \quad \langle x, \pi(e_i) \rangle \leq \lambda_i, \forall i \\
&\Longleftrightarrow \quad \langle x, v_i \rangle \leq \lambda_i, \forall i \\
&\Longleftrightarrow \quad x \in \Delta \ .
\end{aligned}$$

Thus, $y \in \phi(Z) \iff y \in \pi^*(\Delta) = \Delta'$. □

Since we have that Δ' is compact, that ϕ is a proper map and that $\phi(Z) = \Delta'$, we conclude that Z must be bounded, and hence compact.

It remains to show that N acts freely on Z.

Pick a vertex p of Δ, and let $I = \{i_1, \ldots, i_n\}$ be the set of indices for the n facets meeting at p. Pick $z \in Z$ such that $\phi(z) = \pi^*(p)$. Then p is characterized by n equations $\langle p, v_i \rangle = \lambda_i$ where i ranges in I:

$$\begin{aligned}
\langle p, v_i \rangle = \lambda_i \quad &\Longleftrightarrow \quad \langle p, \pi(e_i) \rangle = \lambda_i \\
&\Longleftrightarrow \quad \langle \pi^*(p), e_i \rangle = \lambda_i \\
&\Longleftrightarrow \quad \langle \phi(z), e_i \rangle = \lambda_i \\
&\Longleftrightarrow \quad i\text{-th coordinate of } \phi(z) \text{ is equal to } \lambda_i \\
&\Longleftrightarrow \quad -\tfrac{1}{2}|z_i|^2 + \lambda_i = \lambda_i \\
&\Longleftrightarrow \quad z_i = 0 \ .
\end{aligned}$$

Hence, those z's are points whose coordinates in the set I are zero, and whose other coordinates are nonzero. Without loss of generality, we may assume that $I = \{1, \ldots, n\}$. The stabilizer of z is

$$(\mathbb{T}^d)_z = \{(t_1, \ldots, t_n, 1, \ldots, 1) \in \mathbb{T}^d\} \ .$$

As the restriction $\pi : (\mathbb{R}^d)_z \to \mathbb{R}^n$ maps the vectors $e_1, \ldots, e_n$ to a $\mathbb{Z}$-basis $v_1, \ldots, v_n$ of $\mathbb{Z}^n$ (respectively), at the level of groups, $\pi : (\mathbb{T}^d)_z \to \mathbb{T}^n$ must be bijective. Since $N = \ker(\pi : \mathbb{T}^d \to \mathbb{T}^n)$, we conclude that $N \cap (\mathbb{T}^d)_z = \{e\}$, i.e., $N_z = \{e\}$. Hence all N-stabilizers at points mapping to vertices are trivial. But this was the worst case, since other stabilizers $N_{z'}$ $(z' \in Z)$ are contained in stabilizers for points z which map to vertices. This concludes the proof of Claim 1. □

Given a Delzant polytope Δ, we have constructed a symplectic manifold $(M_\Delta, \omega_\Delta)$ where $M_\Delta = Z/N$ is a compact $2n$-dimensional manifold and ω_Δ is the reduced symplectic form.

Claim 2. The manifold $(M_\Delta, \omega_\Delta)$ is a hamiltonian $\mathbb{T}^n$-space with a moment map μ_Δ having image $\mu_\Delta(M_\Delta) = \Delta$.

Proof of Claim 2. Let z be such that $\phi(z) = \pi^*(p)$ where p is a vertex of Δ, as in the proof of Claim 1. Let $\sigma : \mathbb{T}^n \to (\mathbb{T}^d)_z$ be the inverse for the earlier bijection $\pi : (\mathbb{T}^d)_z \to \mathbb{T}^n$. Since we have found a *section*, i.e., a right inverse for π, in the exact sequence

$$1 \longrightarrow N \xrightarrow{i} \mathbb{T}^d \underset{\overset{\longleftarrow}{\sigma}}{\xrightarrow{\pi}} \mathbb{T}^n \longrightarrow 1 ,$$

the exact sequence *splits*, i.e., becomes like a sequence for a product, as we obtain an isomorphism

$$(i, \sigma) : N \times \mathbb{T}^n \xrightarrow{\simeq} \mathbb{T}^d .$$

The action of the $\mathbb{T}^n$ factor (or, more rigorously, $\sigma(\mathbb{T}^n) \subset \mathbb{T}^d$) descends to the quotient $M_\Delta = Z/N$.

It remains to show that the $\mathbb{T}^n$-action on M_Δ is hamiltonian with appropriate moment map.

Consider the diagram

$$\begin{array}{l} Z \overset{j}{\hookrightarrow} \mathbb{C}^d \xrightarrow{\phi} (\mathbb{R}^d)^* \simeq \eta^* \oplus (\mathbb{R}^n)^* \xrightarrow{\sigma^*} (\mathbb{R}^n)^* \\ p \downarrow \\ M_\Delta \end{array}$$

where the last horizontal map is simply projection onto the second factor. Since the composition of the horizontal maps is constant along N-orbits, it descends to a map

$$\mu_\Delta : M_\Delta \longrightarrow (\mathbb{R}^n)^*$$

which satisfies

$$\mu_\Delta \circ p = \sigma^* \circ \phi \circ j .$$

By Exercise 15 on reduction for product groups, this is a moment map for the action of $\mathbb{T}^n$ on $(M_\Delta, \omega_\Delta)$. Finally, the image of μ_Δ is:

$$\mu_\Delta(M_\Delta) = (\mu_\Delta \circ p)(Z) = (\sigma^* \circ \phi \circ j)(Z) = (\sigma^* \circ \pi^*)(\Delta) = \Delta ,$$

because $\phi(Z) = \pi^*(\Delta)$ and $\sigma^* \circ \pi^* = (\pi \circ \sigma)^* = \mathrm{id}$.

We conclude that $(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu_\Delta)$ is the required toric manifold corresponding to Δ. □

Exercise 16
Let Δ be an n-dimensional Delzant polytope, and let $(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu_\Delta)$ be the associated symplectic toric manifold. Show that μ_Δ maps the fixed points of $\mathbb{T}^n$ bijectively onto the vertices of Δ.

By the remark in Section 2.3, Delzant's construction yields a natural Kähler structure on each symplectic toric manifold.

2.6 Idea Behind Delzant's Construction

The space $\mathbb{R}^d$ is *universal* in the sense that any n-dimensional polytope Δ with d facets can be obtained by intersecting the negative orthant $\mathbb{R}^d_-$ with an affine plane A. Given Δ, to construct A first write Δ as:

$$\Delta = \{x \in \mathbb{R}^n \mid \langle x, v_i \rangle \le \lambda_i,\ i = 1, \ldots, d\} \ .$$

Define

$$\begin{array}{rccc} \pi : & \mathbb{R}^d & \longrightarrow & \mathbb{R}^n \\ & e_i & \longmapsto & v_i \end{array} \qquad \text{with dual map} \qquad \pi^* : \ \mathbb{R}^n \longrightarrow \mathbb{R}^d \ .$$

Then $\pi^* - \lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^d$ is an affine map, where $\lambda = (\lambda_1, \ldots, \lambda_d)$. Let A be the image of $\pi^* - \lambda$. Then A is an n-dimensional affine space.

Lemma 2.6.1. *We have the equality* $(\pi^* - \lambda)(\Delta) = \mathbb{R}^d_- \cap A$.

Proof. Let $x \in \mathbb{R}^n$. Then

$$\begin{array}{rcl} (\pi^* - \lambda)(x) \in \mathbb{R}^d_- & \Longleftrightarrow & \langle \pi^*(x) - \lambda, e_i \rangle \le 0, \forall i \\ & \Longleftrightarrow & \langle x, \pi(e_i) \rangle - \lambda_i \le 0, \forall i \\ & \Longleftrightarrow & \langle x, v_i \rangle \le \lambda_i, \forall i \\ & \Longleftrightarrow & x \in \Delta \ . \end{array}$$

□

We conclude that $\Delta \simeq \mathbb{R}^d_- \cap A$. Now $\mathbb{R}^d_-$ is the image of the moment map for the standard hamiltonian action of $\mathbb{T}^d$ on $\mathbb{C}^d$

$$\begin{array}{rcl} \phi : \mathbb{C}^d & \longrightarrow & \mathbb{R}^d \\ (z_1, \ldots, z_d) & \longmapsto & -\frac{1}{2}(|z_1|^2, \ldots, |z_d|^2) \ . \end{array}$$

Facts.

- The set $\phi^{-1}(A) \subset \mathbb{C}^d$ is a compact submanifold. Let $i : \phi^{-1}(A) \hookrightarrow \mathbb{C}^d$ denote inclusion. Then $i^*\omega_0$ is a closed 2-form which is degenerate. Its kernel is an integrable distribution. The corresponding foliation is called the **null foliation**.

- The null foliation of $i^*\omega_0$ is a principal fibration, so we take the quotient:

$$\begin{array}{ccccc} N & \hookrightarrow & \phi^{-1}(A) & & \\ & & \downarrow & & \\ & & M_\Delta & := & \phi^{-1}(A)/N \ . \end{array}$$

Let ω_Δ be the reduced symplectic form.

- The (non-effective) action of $\mathbb{T}^d = N \times \mathbb{T}^n$ on $\phi^{-1}(A)$ has a "moment map" with image $\phi(\phi^{-1}(A)) = \Delta$. (By "moment map" we mean a map satisfying the usual definition even though the closed 2-form is not symplectic.)

There is a remaining action of $\mathbb{T}^n$ on M_Δ which is hamiltonian with a moment map $\mu_\Delta : M_\Delta \to \mathbb{R}^n$ defined by the commutative diagram

$$\begin{array}{ccccc} \phi^{-1}(A) & \overset{j}{\hookrightarrow} & \mathbb{C}^d & \overset{\phi}{\longrightarrow} & \mathbb{R}^d \\ p\downarrow & & & & \downarrow \mathrm{pr}_2 \\ M_\Delta & & \overset{\mu_\Delta}{\dashrightarrow} & & \mathbb{R}^n \end{array}$$

where $\mathrm{pr}_2 : \mathbb{T}^d = N \times \mathbb{T}^n \to \mathbb{T}^n$ is projection onto the second factor.

Theorem 2.6.2. *For any $x \in \Delta$, we have that $\mu_\Delta^{-1}(x)$ is a single $\mathbb{T}^n$-orbit.*

Proof. Exercise: First consider the standard $\mathbb{T}^d$-action on $\mathbb{C}^d$ with moment map $\phi : \mathbb{C}^d \to \mathbb{R}^d$. Show that $\phi^{-1}(y)$ is a single $\mathbb{T}^d$-orbit for any $y \in \phi(\mathbb{C}^d)$. Now observe that

$$y \in \Delta' = \pi^*(\Delta) \iff \phi^{-1}(y) \subseteq Z \ .$$

Suppose that $y = \pi^*(x)$. Show that $\mu_\Delta^{-1}(x) = \phi^{-1}(y)/N$. But $\phi^{-1}(y)$ is a single $\mathbb{T}^d$-orbit where $\mathbb{T}^d = N \times \mathbb{T}^n$, hence $\mu_\Delta^{-1}(x)$ is a single $\mathbb{T}^n$-orbit. $\square$

Therefore, for toric manifolds, Δ is the orbit space.

Now Δ is a *manifold with corners*. At every point p in a face F, the tangent space $T_p\Delta$ is the subspace of $\mathbb{R}^n$ tangent to F. We can visualize $(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu_\Delta)$ from Δ as follows. First take the product $\mathbb{T}^n \times \Delta$. Let p lie in the interior of $\mathbb{T}^n \times \Delta$. The tangent space at p is $\mathbb{R}^n \times (\mathbb{R}^n)^*$. Define ω_p by:

$$\omega_p(v, \xi) = \xi(v) = -\omega_p(\xi, v) \quad \text{and} \quad \omega_p(v, v') = \omega(\xi, \xi') = 0 \ .$$

for all $v, v' \in \mathbb{R}^n$ and $\xi, \xi' \in (\mathbb{R}^n)^*$. Then ω is a closed nondegenerate 2-form on the interior of $\mathbb{T}^n \times \Delta$. At the corner there are directions missing in $(\mathbb{R}^n)^*$, so ω is a degenerate pairing. Hence, we need to eliminate the corresponding directions in $\mathbb{R}^n$. To do this, we collapse the orbits corresponding to subgroups of $\mathbb{T}^n$ generated by directions orthogonal to the annihilator of that face.

Example. Consider

$$(S^2, \omega = d\theta \wedge dh, S^1, \mu = h) \ ,$$

where S^1 acts on S^2 by rotation. The image of μ is the line segment $I = [-1, 1]$. The product $S^1 \times I$ is an open-ended cylinder. By collapsing each end of the cylinder to a point, we recover the 2-sphere. $\diamondsuit$

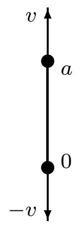
Exercise 17
Build $\mathbb{P}^2$ from $\mathbb{T}^2 \times \Delta$ where Δ is a right-angled isosceles triangle.

Finally, $\mathbb{T}^n$ acts on $\mathbb{T}^n \times \Delta$ by multiplication on the $\mathbb{T}^n$ factor. The moment map for this action is projection onto the Δ factor.

Exercise 18
Follow through the details of Delzant's construction for the case of $\Delta = [0, a] \subset \mathbb{R}^*$ $(n = 1, d = 2)$. Let $v(= 1)$ be the standard basis vector in $\mathbb{R}$. Then Δ is described by

$$\langle x, -v\rangle \leq 0 \qquad \text{and} \qquad \langle x, v\rangle \leq a \ ,$$

where $v_1 = -v$, $v_2 = v$, $\lambda_1 = 0$ and $\lambda_2 = a$.



The projection

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\pi} & \mathbb{R} \\ e_1 & \longmapsto & -v \\ e_2 & \longmapsto & v \end{array}$$

has kernel equal to the span of $(e_1 + e_2)$, so that N is the diagonal subgroup of $\mathbb{T}^2 = S^1 \times S^1$. The exact sequences become

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \xrightarrow{i} & \mathbb{T}^2 & \xrightarrow{\pi} & S^1 & \longrightarrow & 1 \\ & & t & \longmapsto & (t,t) & & & & \\ & & & & (t_1,t_2) & \longmapsto & t_1^{-1}t_2 & & \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{n} & \xrightarrow{i} & \mathbb{R}^2 & \xrightarrow{\pi} & \mathbb{R} & \longrightarrow & 0 \\ & & x & \longmapsto & (x,x) & & & & \\ & & & & (x_1,x_2) & \longmapsto & x_2 - x_1 & & \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R}^* & \xrightarrow{\pi^*} & (\mathbb{R}^2)^* & \xrightarrow{i^*} & \mathfrak{n}^* & \longrightarrow & 0 \\ & & x & \longmapsto & (-x,x) & & & & \\ & & & & (x_1,x_2) & \longmapsto & x_1 + x_2 \ . & & \end{array}$$

The action of the diagonal subgroup $N = \{(e^{it}, e^{it}) \in S^1 \times S^1\}$ on $\mathbb{C}^2$,

$$(e^{it}, e^{it}) \cdot (z_1, z_2) = (e^{it}z_1, e^{it}z_2) \ ,$$

has moment map

$$(i^* \circ \phi)(z_1, z_2) = -\tfrac{1}{2}(|z_1|^2 + |z_2|^2) + a \ ,$$

with zero-level set

$$(i^* \circ \phi)^{-1}(0) = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 2a\} \ .$$

Hence, the reduced space is a projective space:

$$(i^* \circ \phi)^{-1}(0)/N = \mathbb{P}^1 \ .$$

Exercise 19
Consider the standard $(S^1)^3$-action on $\mathbb{P}^3$:

$$(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2 : e^{i\theta_3} z_3] \ .$$

Exhibit explicitly the subsets of $\mathbb{P}^3$ for which the stabilizer under this action is $\{1\}$, S^1, $(S^1)^2$ and $(S^1)^3$. Show that the images of these subsets under the moment map are the interior, the facets, the edges and the vertices, respectively.

Exercise 20
What would be the classification of symplectic toric manifolds if, instead of the equivalence relation defined in Section 1.6, one considered to be equivalent those $(M_i, \omega_i, \mathbb{T}_i, \mu_i)$, $i = 1, 2$, related by an isomorphism $\lambda : \mathbb{T}_1 \to \mathbb{T}_2$ and a λ-equivariant symplectomorphism $\varphi : M_1 \to M_2$ such that:

(a) the maps μ_1 and $\mu_2 \circ \varphi$ are equal up to a constant?

(b) we have $\mu_1 = \ell \circ \mu_2 \circ \varphi$ for some $\ell \in \mathrm{SL}(n; \mathbb{Z})$?

Exercise 21

(a) Classify all 2-dimensional Delzant polytopes with 3 vertices, i.e., triangles, up to translation, change of scale and the action of $\mathrm{SL}(2; \mathbb{Z})$.

Hint: By a linear transformation in $\mathrm{SL}(2; \mathbb{Z})$, we can make one of the angles in the polytope into a square angle. How are the lengths of the two edges forming that angle related?

(b) Classify all 2-dimensional Delzant polytopes with 4 vertices, up to translation and the action of $\mathrm{SL}(2; \mathbb{Z})$.

Hint: By a linear transformation in $\mathrm{SL}(2; \mathbb{Z})$, we can make one of the angles in the polytope into a square angle. Check that automatically another angle also becomes 90^o.

(c) What are all the 4-dimensional symplectic toric manifolds that have four fixed points?

Exercise 22
Let Δ be the n-simplex in $\mathbb{R}^n$ spanned by the origin and the standard basis vectors $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$. Show that the corresponding symplectic toric manifold is projective space, $M_\Delta = \mathbb{P}^n$.

Exercise 23
Which $2n$-dimensional toric manifolds have exactly $n + 1$ fixed points?

Chapter 3

Moment Polytopes

The general theme behind this and Lecture 6 is *how to understand a toric manifold from its polytope*. After reviewing the basics of Morse theory following [39], we compute the homology of symplectic toric manifolds using Morse theory; an appropriate Morse function is provided by a moment map with respect to a suitable circle subgroup. We go on to describe elementary surgery constructions based on symplectic reduction, which hold in the category of symplectic toric manifolds.

3.1 Equivariant Darboux Theorem

The following two theorems describe standard neighborhoods of fixed points. Their proofs rely on the equivariant version of the Moser trick and may be found in [27].

Theorem 3.1.1. (equivariant Darboux) *Let (M, ω) be a $2n$-dimensional symplectic manifold equipped with a symplectic action of a compact Lie group G, and let q be a fixed point. Then there exists a G-invariant chart $(\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at q and G-equivariant with respect to a linear action of G on $\mathbb{R}^{2n}$ such that*

$$\omega|_{\mathcal{U}} = \sum_{k=1}^{n} dx_k \wedge dy_k \ .$$

A suitable linear action on $\mathbb{R}^{2n}$ is equivalent to the induced action of G on T_qM. In particular, if G is a torus, this linear action is characterized by the weights occuring in the representation of G on T_qM. Now any symplectic action is locally hamiltonian. In order to prepare the computation of the Betti numbers of a symplectic toric manifold by using a moment map as a Morse function, we next specify the local picture for a moment map near a fixed point of a hamiltonian torus action.

Theorem 3.1.2. *Let $(M^{2n}, \omega, \mathbb{T}^m, \mu)$ be a hamiltonian $\mathbb{T}^m$-space, where q is a fixed point. Then there exists a chart $(\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at q and weights $\lambda^{(1)}, \ldots, \lambda^{(n)} \in \mathbb{Z}^m$ such that*

(a)

$$\omega|_{\mathcal{U}} = \sum_{k=1}^{n} dx_k \wedge dy_k \ , \qquad \textit{and}$$

(b)

$$\mu|_{\mathcal{U}} = \mu(q) - \frac{1}{2} \sum_{k=1}^{n} \lambda^{(k)} (x_k^2 + y_k^2) \ .$$

This theorem guarantees the existence of a Darboux chart centered at any fixed point where the moment map looks like the moment map for a linear action on $\mathbb{R}^{2n}$. In other words, the real analogue of the model in Exercise 9 is a general local picture near a fixed point of a hamiltonian torus action.

Exercise 24
Show that for a symplectic toric manifold the weights $\lambda^{(1)}, \ldots, \lambda^{(n)}$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^m$.

As a consequence of Theorem 3.1.2 and of the previous exercise, the bijection claimed in Delzant's theorem is well-defined. Indeed, each vertex of a moment polytope satisfies the simplicity, rationality and smoothness conditions which characterize Delzant polytopes.

Another consequence of Theorem 3.1.2 is that a moment map for a symplectic toric manifold yields a lot of *Morse functions*, as we will next explore.

3.2 Morse Theory

Let M be an m-dimensional manifold. A smooth function $f : M \to \mathbb{R}$ is a **Morse function** on M if all of its critical points are nondegenerate.[1]

[1] A point $q \in M$ is a **critical point** of f if $df_q = 0$. A critical point is **nondegenerate** if the **hessian matrix**

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_q$$

is nonsingular, where the x_i's are local coordinates near q. (The condition that the hessian matrix is nonsingular is independent of the choice of coordinates.) The hessian matrix defines a symmetric bilinear function $H_q : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ given by inner product

$$(v, w) \longmapsto \langle v, \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_q w \rangle$$

and also called the **hessian of f at** q relative to the local coordinates x_i; the hessian is in fact the expression in coordinates of a natural bilinear form on the tangent space at q.

The **index** of a bilinear function $H : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is the maximal dimension of a subspace of $\mathbb{R}$ where H is negative definite. The **nullity** of H is the dimension of its nullspace, that is, the subspace consisting of all $v \in \mathbb{R}^m$ such that $H(v, w) = 0$ for all $w \in \mathbb{R}^m$. Hence, a critical point q of $f : M \to \mathbb{R}$ is nondegenerate if and only if the hessian $H_q : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ has nullity equal to zero.

Let q be a nondegenerate critical point for $f : M \to \mathbb{R}$. The **index of f at q** is the index of the hessian $H_q : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$. This is well-defined, i.e., the index is independent of the choice of local coordinates. Moreover, the **Morse lemma** states that there is a coordinate chart $(\mathcal{U}, x_1, \ldots, x_m)$ centered at q such that

$$f|_{\mathcal{U}} = f(q) - (x_1)^2 - \ldots - (x_\lambda)^2 + (x_{\lambda+1})^2 + \ldots + (x_m)^2 \ ,$$

where λ is the index of f at q. In particular, nondegenerate critical points are necessarily isolated.

Let f be a Morse function on M. For $a \in \mathbb{R}$, let

$$M^a = f^{-1}(-\infty, a] = \{p \in M \mid f(p) \leq a\} \ .$$

Theorem 3.2.1. (Morse [40], Milnor [39])

(a) Let $a < b$ and suppose that the set $f^{-1}[a, b]$, consisting of all $p \in M$ with $a \leq f(p) \leq b$, is compact, and contains no critical points of f. Then M^a is diffeomorphic to M^b. Furthermore, M^a is a deformation retract of M^b, so that the inclusion map $M^a \hookrightarrow M^b$ is a homotopy equivalence.

(b) Let q be a nondegenerate critical point with index λ and $f(q) = c$. Suppose that $f^{-1}[c - \varepsilon, c + \varepsilon]$ is compact, and contains no critical point of f other than q, for some $\varepsilon > 0$. Then, for all sufficiently small ε, the set $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with a λ-cell attached.

(c) If each set M^a is compact, then the manifold M has the homotopy type of a CW-complex with one cell of dimension λ for each critical point of index λ.

A k-cell is simply a k-dimensional disk D^k, and it gets attached along its boundary S^{k-1}. Morse's original treatment did not include part (c) of Theorem 3.2.1. Instead, his main results were phrased in terms of inequalities. Let $b_k(M) := \dim H_k(M)$ be the **k-th Betti number of** M. Let M be a compact manifold and f a Morse function on M. Let C_λ be the number of critical points of f with index λ.

Theorem 3.2.2. (Morse inequalities [40])

(a)

$$b_\lambda(M) \leq C_\lambda \ ,$$

(b)

$$\sum_\lambda (-1)^\lambda b_\lambda(M) = \sum_\lambda (-1)^\lambda C_\lambda \ , \ \textit{and}$$

(c)

$$b_\lambda(M) - b_{\lambda-1}(M) + \ldots \pm b_0(M) \leq C_\lambda - C_{\lambda-1} + \ldots \pm C_0 \ .$$

A **perfect Morse function** is a Morse function for which the inequalities in the previous statement are equalities.

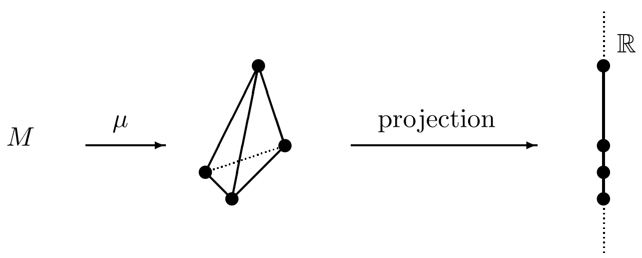
Corollary 3.2.3. *If all critical points of a Morse function f have even index, then f is a perfect Morse function.*

3.3 Homology of Symplectic Toric Manifolds

Let $(M, \omega, \mathbb{T}^n, \mu)$ be a $2n$-dimensional symplectic toric manifold. Choose a suitably generic direction in $\mathbb{R}^n$ by picking a vector X whose components are independent over $\mathbb{Q}$. This condition ensures that:

- the one-dimensional subgroup, $T^X \subset \mathbb{T}^n$, generated by the vector X is dense in $\mathbb{T}^n$,
- X is not parallel to the facets of the moment polytope $\Delta := \mu(M)$, and
- the vertices of Δ have different projections along X.

Exercise 25
Check that the fixed points for the $\mathbb{T}^n$-action are exactly the fixed points of the action restricted to $\mathbb{T}^X$, that is, are the zeros of the vector field, $X^\#$ on M corresponding to the $\mathbb{T}^X$-action.



Let $\mu^X := \langle \mu, X \rangle : M \to \mathbb{R}$ be the projection of μ along X. By definition of moment map, μ^X is a hamiltonian function for the vector field $X^\#$ generated by X. We conclude from the previous exercise that the critical points of μ^X are precisely the fixed points of the $\mathbb{T}^n$-action.

By Theorem 3.1.2, if q is a fixed point for the $\mathbb{T}^n$-action, then there exists a chart $(\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at q and weights $\lambda^{(1)}, \ldots, \lambda^{(n)} \in \mathbb{Z}^n$ such that

$$\mu^X\big|_{\mathcal{U}} = \langle \mu, X\rangle|_{\mathcal{U}} = \mu^X(q) - \frac{1}{2}\sum_{k=1}^{n} \langle \lambda^{(k)}, X\rangle (x_k^2 + y_k^2) \ .$$

Since the components of X are independent over $\mathbb{Q}$, all coefficients $\langle \lambda^{(k)}, X\rangle$ are nonzero, so q is a *nondegenerate* critical point of μ^X. Moreover, the index of q is twice the number of labels k such that $-\langle \lambda^{(k)}, X\rangle < 0$. But the $-\lambda^{(k)}$'s are precisely the edge vectors u_i which satisfy Delzant's conditions. Therefore, geometrically, the index of q can be read from the moment polytope Δ, by taking twice the number of edges whose inward-pointing edge vectors at $\mu(q)$ *point up relative to* X, that is, whose inner product with X is positive. In particular, μ^X is a perfect Morse function. By applying Corollary 3.2.3 we conclude that:

Theorem 3.3.1. *Let $X \in \mathbb{R}^n$ have components independent over $\mathbb{Q}$. The degree-$2k$ homology group of the symplectic toric manifold $(M, \omega, \mathbb{T}, \mu)$ has dimension equal to the number of vertices of the moment polytope Δ where there are exactly k (primitive inward-pointing) edge vectors which point up relative to the projection along the X. All odd-degree homology groups of M are zero.*

Of course, by Poincaré duality (or by taking $-X$ instead of X), the words "point up" may be replaced by "point down".

Exercise 26
Let $(M, \omega, \mathbb{T}, \mu)$ be a symplectic toric manifold. What is the Euler characteristic of M?

3.4 Symplectic Blow-Up

Let L be the tautological line bundle over $\mathbb{P}^{n-1}$, that is,

$$L = \{([p], z) \mid p \in \mathbb{C}^n \setminus \{0\} \ , \ z = \lambda p \text{ for some } \lambda \in \mathbb{C}\}$$

with projection to $\mathbb{P}^{n-1}$ given by $([p], z) \mapsto [p]$. The fiber of L over the point $[p] \in \mathbb{P}^{n-1}$ is the complex line in $\mathbb{C}^n$ represented by that point.

Definition 3.4.1. *The* **blow-up of $\mathbb{C}^n$ at the origin** *is the total space of the bundle L. The corresponding* **blow-down map** *is the map $\beta : L \to \mathbb{C}^n$ defined by $\beta([p], z) = z$.*

Notice that the total space of L may be decomposed as the disjoint union of two sets,

$$E := \{([p], 0) \mid p \in \mathbb{C}^n \setminus \{0\}\}$$

and

$$S := \{([p], z) \mid p \in \mathbb{C}^n \setminus \{0\} \ , \ z = \lambda p \text{ for some } \lambda \in \mathbb{C}^*\} \ .$$

The set E is called the **exceptional divisor**; it is diffeomorphic to $\mathbb{P}^{n-1}$ and gets mapped to the origin by β. On the other hand, the restriction of β to the complementary set S is a diffeomorphism onto $\mathbb{C}^n \setminus \{0\}$. Hence, we may regard L as being obtained from $\mathbb{C}^n$ by smoothly replacing the origin by a copy of $\mathbb{P}^{n-1}$.

There are actions of the unitary group $\mathrm{U}(n)$ on all of these sets induced by the standard linear action on $\mathbb{C}^n$, and the map β is $\mathrm{U}(n)$-equivariant.

Definition 3.4.2. *A* **blow-up symplectic form** *on L is a $\mathrm{U}(n)$-invariant symplectic form ω such that the difference $\omega - \beta^*\omega_0$ is compactly supported, where $\omega_0 = \frac{i}{2}\sum_{k=1}^{n} dz_k \wedge d\bar{z}_k$ is the standard symplectic form on $\mathbb{C}^n$.*

Two blow-up symplectic forms are called **equivalent** if one is the pullback of the other by a $\mathrm{U}(n)$-equivariant diffeomorphism of L. Guillemin and Sternberg [26] have shown that two blow-up symplectic forms are equivalent if and only if they have equal restrictions to the exceptional divisor $E \subset L$.

Let Ω^ε ($\varepsilon > 0$) be the set of all blow-up symplectic forms on L whose restriction to the exceptional divisor $E \simeq \mathbb{P}^{n-1}$ is $\varepsilon\omega_{\mathrm{FS}}$, where ω_{FS} is the Fubini-Study form on $\mathbb{P}^{n-1}$ described in Lecture 2. An **ε-blow-up** of $\mathbb{C}^n$ at the origin is a pair (L, ω) with $\omega \in \Omega^\varepsilon$.

Let (M, ω) be a $2n$-dimensional symplectic manifold. It is a consequence of the Darboux theorem that, for each point $q \in M$, there exists a chart $(\mathcal{U}, z_1, \ldots, z_n)$ centered at q and with image in $\mathbb{C}^n$ where

$$\omega|_{\mathcal{U}} = \frac{i}{2}\sum_{k=1}^{n} dz_k \wedge d\bar{z}_k \ .$$

It is shown in [26] that, for ε small enough, we can perform an ε-blow-up of M at q modeled on $\mathbb{C}^n$ at the origin, without changing the symplectic structure outside of a small neighborhood of q. The resulting manifold is then called an **ε-blow-up of M at q**.

Example. Let $\mathbb{P}(L \oplus \mathbb{C})$ be the $\mathbb{P}^1$-bundle over $\mathbb{P}^{n-1}$ obtained by projectivizing the direct sum of the tautological line bundle L with a trivial complex line bundle. Consider the map

$$\begin{array}{rrcl} \beta : & \mathbb{P}(L \oplus \mathbb{C}) & \longrightarrow & \mathbb{P}^n \\ & ([p], [\lambda p : w]) & \longmapsto & [\lambda p : w] \ , \end{array}$$

where $[\lambda p : w]$ on the right represents a line in $\mathbb{C}^{n+1}$, forgetting that, for each $[p] \in \mathbb{P}^{n-1}$, that line sits in the 2-complex-dimensional subspace $L_{[p]} \oplus \mathbb{C} \subset \mathbb{C}^n \oplus \mathbb{C}$. Notice that β maps the *exceptional divisor*

$$E := \{([p], [0 : \ldots : 0 : 1]) \mid [p] \in \mathbb{P}^{n-1}\} \simeq \mathbb{P}^{n-1}$$

to the point $[0 : \ldots : 0 : 1] \in \mathbb{P}^n$, whereas β is a diffeomorphism on the complement

$$S := \{([p], [\lambda p : w]) \mid [p] \in \mathbb{P}^{n-1} \ , \ \lambda \in \mathbb{C}^* \ , \ w \in \mathbb{C}\} \simeq \mathbb{P}^n \setminus \{[0 : \ldots : 0 : 1]\} \ .$$

Therefore, we may regard $\mathbb{P}(L \oplus \mathbb{C})$ as being obtained from $\mathbb{P}^n$ by smoothly replacing the point $[0 : \ldots : 0 : 1]$ by a copy of $\mathbb{P}^{n-1}$. The space $\mathbb{P}(L \oplus \mathbb{C})$ is the blow-up of $\mathbb{P}^n$ at the point $[0 : \ldots : 0 : 1]$, and β is the corresponding blow-down map. The manifold $\mathbb{P}(L \oplus \mathbb{C})$ for $n = 2$ is known as the **first Hirzebruch surface**. $\diamondsuit$

Exercise 27
Write a definition for *blow-up of a symplectic manifold along a complex submanifold* by considering the projectivization of the normal bundle to the submanifold.

Symplectic blow-up is due to Gromov according to the first printed exposition of this operation in [35].

3.5 Blow-Up of Toric Manifolds

Suppose that a compact Lie group acts on a symplectic manifold (M, ω) in a hamiltonian way, and that $q \in M$ is a fixed point for the G-action. Then, by Theorem 3.1.1, there exists a Darboux chart $(\mathcal{U}, z_1, \ldots, z_n)$ centered at q which is G-equivariant with respect to a linear action of G on $\mathbb{C}^n$. Consider an ε-blow-up of M relative to this chart, for ε sufficiently small.

Exercise 28
Check that G acts on the blow-up in a hamiltonian way. Describe the moment map.

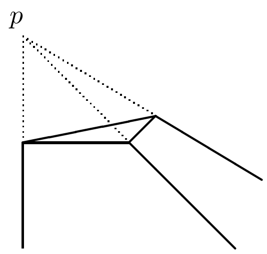
Let Δ be an n-dimensional Delzant polytope, and let $(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu_\Delta)$ be the associated symplectic toric manifold. The ε-blow-up of $(M_\Delta, \omega_\Delta)$ at a fixed point of the $\mathbb{T}^n$-action is a new symplectic toric manifold. What is the moment polytope Δ_ε corresponding to this new symplectic toric manifold?

Let q be a fixed point of the $\mathbb{T}^n$-action on $(M_\Delta, \omega_\Delta)$, and let $p = \mu_\Delta(q)$ be the corresponding vertex of Δ. (Cf. Exercise 16.) Let $u_1, \ldots, u_n$ be the primitive (inward-pointing) edge vectors at p, so that the rays $p + tu_i$, $t \geq 0$, form the edges of Δ at p.

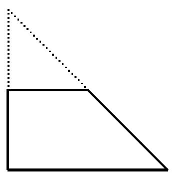
Theorem 3.5.1. *The ε-blow-up of $(M_\Delta, \omega_\Delta)$ at a fixed point q is the symplectic toric manifold associated to the polytope Δ_ε obtained from Δ by replacing the vertex p by the n vertices*

$$p + \varepsilon u_i \ , \quad i = 1, \ldots, n \ .$$

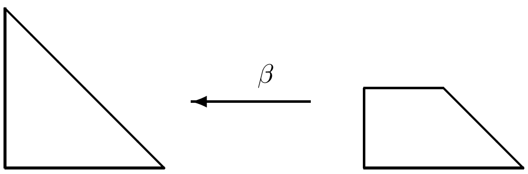
In other words, the moment polytope for the blow-up of $(M_\Delta, \omega_\Delta)$ at q is obtained from Δ by chopping off the corner corresponding to q, thus substituting the original set of vertices by the same set with the vertex corresponding to q replaced by exactly n new vertices:



Proof. Exercise: Check that the new polytope is Delzant. We may view the ε-blow-up of $(M_\Delta, \omega_\Delta)$ as being obtained from M_Δ by smoothly replacing q by $(\mathbb{P}^{n-1}, \varepsilon\omega_{\text{FS}})$. Compute the restriction of the moment map to this set. Recall Exercise 22. □



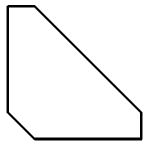
Example. The moment polytope for the standard $\mathbb{T}^2$-action on $(\mathbb{P}^2, \omega_{\text{FS}})$ is a right isosceles triangle Δ. If we blow-up $\mathbb{P}^2$ at $[0 : 0 : 1]$ we obtain a symplectic toric manifold associated to the trapezoid below.



Exercise 29
Check that this manifold is the first Hirzebruch surface, defined in Section 3.4.

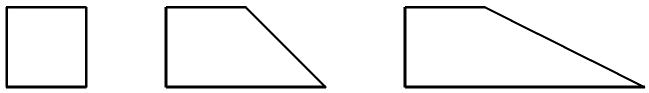
◇

Example. The following moment polytope corresponds to a toric manifold obtained by blowing-up $\mathbb{P}^2$ at the three fixed points:



$\diamondsuit$

Exercise 30
The toric 4-manifold $\mathcal{H}_n$ corresponding to the polygon with vertices $(0,0)$, $(n+1,0)$, $(0,1)$ and $(1,1)$, for n a nonnegative integer, is called a **Hirzebruch surface**.



The manifold $\mathcal{H}_0$ is just a product of two spheres, whereas by the previous example $\mathcal{H}_1$ is a blow-up of $\mathbb{P}^2$ at a point.

(a) Construct the manifold $\mathcal{H}_n$ by symplectic reduction of $\mathbb{C}^4$ with respect to an action of $(S^1)^2$.

(b) Exhibit $\mathcal{H}_n$ as a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.

3.6 Symplectic Cutting

Let (M,ω) be a symplectic manifold where S^1 acts in a hamiltonian way, $\rho : S^1 \to \mathrm{Diff}(M)$, with moment map $\mu : M \to \mathbb{R}$. Suppose that:

- M has a unique nondegenerate minimum at q where $\mu(q) = 0$, and
- for ε sufficiently small, S^1 acts freely on the level set $\mu^{-1}(\varepsilon)$.

Let $\mathbb{C}$ be equipped with the symplectic form $-idz \wedge d\bar{z}$. Then the action of S^1 on the product

$$\psi : S^1 \longrightarrow \mathrm{Diff}(M \times \mathbb{C}) \ , \qquad \psi_t(p,z) = (\rho_t(p), t \cdot z) \ ,$$

is hamiltonian with moment map

$$\phi : M \times \mathbb{C} \longrightarrow \mathbb{R} \ , \qquad \phi(p,z) = \mu(p) - |z|^2 \ .$$

Observe that S^1 acts freely on the ε-level of ϕ for ε small enough:

$$\begin{array}{rcl}\phi^{-1}(\varepsilon) & = & \{(p,z) \in M \times \mathbb{C} \mid \mu(p) - |z|^2 = \varepsilon\} \\ & = & \{(p,0) \in M \times \mathbb{C} \mid \mu(p) = \varepsilon\} \\ & & \cup \quad \{(p,z) \in M \times \mathbb{C} \mid |z|^2 = \mu(p) - \varepsilon > 0\} \ .\end{array}$$

The reduced space is hence

$$\phi^{-1}(\varepsilon)/S^1 \simeq \mu^{-1}(\varepsilon)/S^1 \cup \{p \in M \mid \mu(p) > \varepsilon\} \ .$$

The open submanifold of M given by $\{p \in M \mid \mu(p) > \varepsilon\}$ embeds as an open dense submanifold into $\phi^{-1}(\varepsilon)/S^1$.

Exercise 31
Show that the reduced space $\phi^{-1}(\varepsilon)/S^1$ is the ε-blow-up of M at q.

This global description of blow-up for hamiltonian S^1-spaces is due to Lerman [31], as a particular instance of his *cutting* technique. **Symplectic cutting** is the application of symplectic reduction to the product of a hamiltonian S^1-space with the standard $\mathbb{C}$ as above, in a way that the reduced space for the original hamiltonian S^1-space embeds symplectically as a codimension 2 submanifold in a symplectic manifold.

As it is a local construction, the cutting operation may be more generally performed at a local minimum (or maximum) of the moment map μ.

There is a remaining S^1-action on the cut space $M^{\geq\varepsilon}_{\text{cut}} := \phi^{-1}(\varepsilon)/S^1$ induced by

$$\tau : S^1 \longrightarrow \operatorname{Diff}(M \times \mathbb{C}) \ , \qquad \tau_t(p,z) = (\rho_t(p), z) \ .$$

In fact, τ is a hamiltonian S^1-action on $M \times \mathbb{C}$ which commutes with ψ, thus descends to an action $\widetilde{\tau} : S^1 \to \operatorname{Diff}(M^{\geq\varepsilon}_{\text{cut}})$.

Exercise 32
Show that $\widetilde{\tau}$ is hamiltonian by describing a moment map.

Loosely speaking, the cutting technique provides a hamiltonian way to close the open manifold $\{p \in M \mid \mu(p) > \varepsilon\}$, by using the reduced space at level ε, $\mu^{-1}(\varepsilon)/S^1$. We may similarly close $\{p \in M \mid \mu(p) < \varepsilon\}$. The resulting hamiltonian S^1-spaces are called **cut spaces**, and denoted $M^{\geq\varepsilon}_{\text{cut}}$ and $M^{\leq\varepsilon}_{\text{cut}}$. Of course, if another group G acts on M in a hamiltonian way which commutes with the S^1-action, then the cut spaces are also hamiltonian G-spaces.

Part II

Algebraic Viewpoint

Chapter 4

Toric Varieties

The goal of this lecture is to explain toric manifolds as a special class of projective varieties. The first five sections contain a crash course on notions and basic facts about algebraic varieties, mostly in order to fix notation. The combinatorial flavor of toric varieties is postponed until Lecture 5.

4.1 Affine Varieties

Let $\mathbb{C}[z_1, \ldots, z_n]$ be the algebra of polynomials in the n complex coordinate functions on $\mathbb{C}^n$. Throughout this lecture, we consider the **Zariski topology** on $\mathbb{C}^n$: a **(Zariski) closed set** in $\mathbb{C}^n$ is a set of common zeros of a finite number of polynomials from $\mathbb{C}[z_1, \ldots, z_n]$; naturally, the complement of a Zariski closed set is called a **(Zariski) open set**. The fact that any infinite intersection of closed sets is indeed a closed set follows from the stabilization property for sets given as zero sets of polynomials: any decreasing sequence of such sets $X_1 \supset X_2 \supset \ldots$ stabilizes, i.e., there exists an integer r such that $X_r = X_{r+1} = \ldots$. This is a restatement of Hilbert's basis theorem which says that any ideal in $\mathbb{C}[z_1, \ldots, z_n]$ is finitely generated; see, for instance, [50, §IV-1]. Notice that any nonempty open set is dense (i.e., its closure is the full space), hence the Zariski topology is not Hausdorff.

Definition 4.1.1. *An* **affine variety** *is a nonempty closed set in a* $\mathbb{C}^n$.

The **Zariski topology on an affine variety** $X \subset \mathbb{C}^n$ is the topology on X which declares to be **(Zariski) open** (respectively, **closed**) every set which is the intersection of X with an open (resp., closed) subset of $\mathbb{C}^n$. An affine variety X is **irreducible** if it cannot be written as the union of two proper closed subsets. On an irreducible affine variety, any nonempty open subset is dense.

Example. The zero locus in $\mathbb{C}^2$ of the polynomial $z_1 z_2$ is not irreducible; its irreducible components are given by the lines $z_1 = 0$ and $z_2 = 0$. ♢

Exercise 33
Let $X \subset \mathbb{C}^n$ be the affine variety defined as the zero locus of polynomials $p_1, \ldots, p_r \in \mathbb{C}[z_1, \ldots, z_n]$. Show that X is irreducible if and only if the ideal generated by $p_1, \ldots, p_r$ is *prime* (an ideal $I \subset \mathbb{C}[z_1, \ldots, z_n]$ is **prime** if whenever $u, v \in \mathbb{C}[z_1, \ldots, z_n]$ and $uv \in I$, then $u \in I$ or $v \in I$).

Let $X \subset \mathbb{C}^n$ be an affine variety presented as the zero locus of polynomials $p_1, \ldots, p_r \in \mathbb{C}[z_1, \ldots, z_n]$.

Definition 4.1.2. *A* **regular function** *on X is a function $X \to \mathbb{C}$ which is the restriction to X of a polynomial function in $\mathbb{C}^n$. The ring of regular functions on X is denoted* $\mathbb{C}[X]$.

Exercise 34
Show that the ring $\mathbb{C}[X]$ is isomorphic to $\mathbb{C}[z_1, \ldots, z_n]/I$, where I is the ideal generated by $p_1, \ldots, p_r$.

Let $X \subset \mathbb{C}^n$ and $X' \subset \mathbb{C}^m$ be affine varieties.

Definition 4.1.3. *A* **regular map** *from X to X' is a map $\varphi : X \to X'$ which is the restriction of a polynomial map $\mathbb{C}^n \to \mathbb{C}^m$.*

Exercise 35
Show that a map $\varphi : X \to X'$ is regular if and only if it pulls back regular functions on X' to regular functions on X.

Definition 4.1.4. *An* **isomorphism** *from X to X' is a regular map $X \to X'$ which is invertible by a regular map. The symbol $\simeq$ indicates an isomorphism. The group of isomorphisms $X \to X$ is denoted by* $\mathrm{Isom}(X)$. *The affine varieties X and X' are* **isomorphic** *when there exists an isomorphism between them.*

Exercise 36
Show that X and X' are isomorphic if and only if the associated rings of regular functions, $\mathbb{C}[X]$ and $\mathbb{C}[X']$, are isomorphic.

Example. Consider the variety $X = \{z_i z_{n+i} = 1 \ , i = 1, \ldots, n\} \subset \mathbb{C}^{2n}$; for $n = 1$, this is the (complex) hyperbola with $\{z_1 = 0\}$ and $\{z_2 = 0\}$ as asymptotes. On X the functions $z_1, \ldots, z_n \in \mathbb{C}[X]$ are invertible by regular functions: $z_i^{-1} = z_{n+i}$. Hence the ring of regular functions on X is the ring of Laurent polynomials in n variables:

$$\mathbb{C}[X] = \mathbb{C}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}] \ .$$

The projection $\mathbb{C}^{2n} \to \mathbb{C}^n$ onto the first n components maps X isomorphically onto the n**-dimensional algebraic torus**

$$(\mathbb{C}^*)^n := (\mathbb{C} \setminus \{0\})^n = \mathbb{C}^n \setminus \text{ hyperplanes } z_i = 0 \ , i = 1, \ldots n \ .$$

Hence we have

$$\begin{array}{rcl} X & \stackrel{\simeq}{\longleftrightarrow} & (\mathbb{C}^*)^n \\ (z_1, \ldots, z_n, z_{n+1}, \ldots, z_{2n}) & \longmapsto & (z_1, \ldots, z_n) \\ (z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}) & \longleftarrow & (z_1, \ldots, z_n) \ . \end{array}$$

This shows that the torus $(\mathbb{C}^*)^n$ is an affine variety. Note that the coordinates $z_1, \ldots, z_n$ are invertible by regular functions on $(\mathbb{C}^*)^n$, whereas they were not invertible by regular functions on $\mathbb{C}^n$. $\diamondsuit$

4.2 Rational Maps on Affine Varieties

If $X \subset \mathbb{C}^n$ and $X' \subset \mathbb{C}^m$ are affine varieties, the symbol $X \dashrightarrow X'$ denotes a map defined on some open subset of X.

Example. If $p_1, p_2 \in \mathbb{C}[z_1, \ldots z_n]$, then the rational function defined on the set $\{p_2 \neq 0\}$ by $z \mapsto \frac{p_1(z)}{p_2(z)}$ is a map $\mathbb{C}^n \dashrightarrow \mathbb{C}$. $\diamondsuit$

Let $X \subset \mathbb{C}^n$ be an irreducible affine variety.

Definition 4.2.1. *A* **rational function** *on X is a map $X \dashrightarrow \mathbb{C}$ which is the restriction of a rational function on $\mathbb{C}^n$ whose denominator does not vanish identically on X. The ring of rational functions on X is denoted $\mathcal{O}_X$.*

Let $\mathbb{C}(X)$ denote the field of fractions of $\mathbb{C}[X]$, and let $\mathcal{I}_X$ be the ideal in $\mathcal{O}_X$ formed by the rational functions on $\mathbb{C}^n$ whose numerator vanishes identically on X.

Exercise 37
Show that the field $\mathbb{C}(X)$ is isomorphic to the quotient $\mathcal{O}_X/\mathcal{I}_X$.

An irreducible affine variety X is **normal** if its ring of regular functions $\mathbb{C}[X]$ is integrally closed in its field of fractions, that is, for any $f \in \mathbb{C}(X)$, if f satisfies an equation of the form

$$f^m + g_1 f^{m-1} + \ldots + g_m = 0$$

with coefficients $g_i \in \mathbb{C}[X]$, then $f \in \mathbb{C}[X]$.

Any smooth variety is normal and the set of singular points of a normal variety has codimension at least 2 [44].

Examples.

1. On the curve $X \subset \mathbb{C}^2$ defined by $y^2 = x^2 + x^3$, the rational function $t = \frac{y}{x} \in \mathbb{C}(X)$ is integral over $\mathbb{C}[X]$ since $t^2 - 1 - x = 0$, but $t \notin \mathbb{C}[X]$, hence X is *not* normal.

2. The cone $X \subset \mathbb{C}^3$ given by $x^2 + y^2 = z^2$ is normal [44], though it has a singular point at the origin

$\diamondsuit$

Let $X \subset \mathbb{C}^n$ and $X' \subset \mathbb{C}^m$ be affine varieties.

Definition 4.2.2. *A* **rational map** *from X to X' is a map $\varphi : X \dashrightarrow X'$ which is the restriction of a rational map $\mathbb{C}^n \dashrightarrow \mathbb{C}^m$ whose denominator does not vanish identically on X.*

Exercise 38
Show that a map $\varphi : X \dashrightarrow X'$ is rational if and only if it pulls back rational functions on X' to rational functions on X.

Definition 4.2.3. *A* **birational equivalence** *from X to X' is a rational map $X \dashrightarrow X'$ which is invertible by a rational map. The affine varieties X and X' are* **birationally equivalent** *when there exists a birational equivalence between them.*

Exercise 39
Show that X and X' are birationally equivalent if and only if the associated fields of rational functions, $\mathbb{C}(X)$ and $\mathbb{C}(X')$, are isomorphic.

A **hypersurface in** $\mathbb{C}^n$ is the zero set of one polynomial in $\mathbb{C}[z_1, \ldots, z_n]$. Any affine variety is birationally equivalent to a hypersurface of some space $\mathbb{C}^m$; for a proof see, for example, [44].

4.3 Projective Varieties

We say that a polynomial $p \in \mathbb{C}[z_0, \ldots, z_n]$ vanishes at a point $[w_0 : \ldots : w_n] \in \mathbb{P}^n$ if $p(\lambda w_0, \ldots, \lambda w_n) = 0$ for all $\lambda \in \mathbb{C}^*$. Notice that this condition implies that each homogeneous component of p vanishes.

We consider the **Zariski topology** on $\mathbb{P}^n$: a **(Zariski) closed set** in $\mathbb{P}^n$ is a set of common zeros of a finite number of polynomials from $\mathbb{C}[z_0, \ldots, z_n]$; as usual, the complement of a Zariski closed set is called a **(Zariski) open set**. By considering homogeneous components, we may assume that each of those polynomials is homogeneous. A **hypersurface in** $\mathbb{P}^n$ is the zero set of one (reduced) homogeneous polynomial in $\mathbb{C}[z_0, \ldots, z_n] \setminus \{0\}$, the degree of which is called the **degree** of the hypersurface[1]. A hypersurface of degree $2, 3, 4, \ldots$ is traditionally called a **quadric** (except in $\mathbb{P}^2$ when it is called a **conic**), a **cubic**, a **quartic**, etc.[2]

[1] A polynomial is **reduced** if each of its irreducible factors has multiplicity 1. For some applications it is convenient to allow for nonreduced polynomials and then consider that some components of the hypersurface have multiplicities.

[2] A hypersurface of degree 1 is called a *line* when in $\mathbb{P}^2$, a *plane* when in $\mathbb{P}^3$, and a *hyperplane* in higher projective spaces.

Definition 4.3.1. *A* **projective variety** *is a nonempty closed subset of some projective space* $\mathbb{P}^n$.

The **Zariski topology on a projective variety** $X \subset \mathbb{P}^n$ is the topology on X which declares to be **(Zariski) open** (respectively, **closed**) every set which is the intersection of X with an open (resp., closed) subset of $\mathbb{P}^n$. A projective variety X is **irreducible** if it cannot be written as the union of two proper closed subsets.

Examples.

1. The product of two projective spaces is a projective variety. This can be seen via the **Segre embedding**

$$\begin{array}{rccl} S: & \mathbb{P}^n \times \mathbb{P}^m & \longrightarrow & \mathbb{P}^{nm+n+m} \\ & ([z],[w]) & \longmapsto & [z \otimes w] \ . \end{array}$$

   The homogeneous coordinates of an image point $[z \otimes w]$ are

$$y_{ij} := z_i w_j \ , \quad i = 0, \ldots, n \ , \ j = 0, \ldots, m \ .$$

   The image of S is the set cut out by the system of equations

$$y_{ij} y_{k\ell} = y_{kj} y_{i\ell} \ , \qquad \begin{cases} i,k = 0,\ldots,n \\ j,\ell = 0,\ldots,m \end{cases}$$

   thus $S(\mathbb{P}^n \times \mathbb{P}^m) \simeq \mathbb{P}^n \times \mathbb{P}^m$ is a nonempty closed subset of $\mathbb{P}^{nm+n+m}$. In particular, $S(\mathbb{P}^1 \times \mathbb{P}^1)$ is the subset of points $[y_{00} : y_{01} : y_{10} : y_{11}] \in \mathbb{P}^3$ determined by the single quadratic equation

$$y_{00} y_{11} = y_{01} y_{10} \ ,$$

   hence $S(\mathbb{P}^1 \times \mathbb{P}^1)$ is a nondegenerate quadric in $\mathbb{P}^3$.

2. The set of lines in $\mathbb{P}^n$ through the point $[0 : \ldots : 0 : 1] \in \mathbb{P}^n$ (or, similarly, through any other point), is naturally identified with $\mathbb{P}^{n-1}$ via

$$\begin{array}{rcl} \mathbb{P}^{n-1} & \longleftrightarrow & \{ \text{ lines in } \mathbb{P}^n \text{ through } [0:\ldots:0:1]\} \\ [w] & \longleftrightarrow & L_{[w]} := \{[\lambda w_0 : \ldots : \lambda w_{n-1} : \gamma] \mid (\lambda,\gamma) \in \mathbb{C}^2 \setminus \{(0,0)\}\} \ . \end{array}$$

   The **blow-up of** $\mathbb{P}^n$ **at** $[0 : \ldots : 0 : 1]$, $B(\mathbb{P}^n, [0 : \ldots : 0 : 1])$, is the subset of $\mathbb{P}^n \times \mathbb{P}^{n-1}$ defined by the incidence relation

$$B(\mathbb{P}^n, [0:\ldots:0:1]) := \{([w],[z]) \in \mathbb{P}^{n-1} \times \mathbb{P}^n \mid [z] \in L_{[w]}\} \ .$$

   This can be translated as the closed subset of $\mathbb{P}^n \times \mathbb{P}^{n-1}$ defined by the system of equations

$$z_i w_j = z_j w_i \ , \quad \forall \ 0 \le i, j \le n-1 \ .$$

The Segre embedding exhibits $\mathbb{P}^n \times \mathbb{P}^{n-1}$ as a projective variety in $\mathbb{P}^{n^2+n-1}$, hence $B(\mathbb{P}^n, [0 : \ldots : 0 : 1])$ is a projective variety.

Since the point $[0 : \ldots : 0 : 1]$ belongs to any line through it, the set $B(\mathbb{P}^n, [0 : \ldots : 0 : 1])$ decomposes as the disjoint union of the so-called *exceptional divisor*,

$$E := \{([w], [0 : \ldots : 0 : 1]) \mid [w] \in \mathbb{P}^{n-1}\} \simeq \mathbb{P}^{n-1} ,$$

with

$$S := \{([w], [w : y]) \mid [w] \in \mathbb{P}^{n-1} , \ y \in \mathbb{C}\} \simeq \mathbb{P}^n \setminus \{[0 : \ldots : 0 : 1]\} .$$

$\diamondsuit$

Let $X \subset \mathbb{P}^n$ be a projective variety.

Definition 4.3.2. *A* **regular function** *on X is a function $X \to \mathbb{C}$ which, locally near each point $x \in X$, may be written as a quotient of two homogeneous polynomials of the same degree such that the denominator does not vanish at x. The ring of regular functions on X is denoted $\mathbb{C}[X]$.*

Any regular function on $\mathbb{P}^n$ is constant. More generally, it can be shown that $\mathbb{C}[X] = \mathbb{C}$ whenever X is an irreducible projective variety [44]. Therefore, the ring $\mathbb{C}[X]$ will not give much information.

Let $X \subset \mathbb{P}^n$ and $X' \subset \mathbb{P}^m$ be projective varieties. Recall that complex projective space $\mathbb{P}^n$ comes equipped with $n + 1$ **affine neighborhoods** given by the standard charts ($k = 0. \ldots , n$):

$$\begin{array}{rcl} \mathcal{V}_k := \{[z_0 : \ldots : z_n] \mid z_k \neq 0\} & \simeq & \mathbb{C}^n \\ {[z_0 : \ldots : z_n]} & \mapsto & \left(\frac{z_0}{z_k}, \ldots , \frac{z_{k-1}}{z_k}, \frac{z_{k+1}}{z_k}, \ldots , \frac{z_n}{z_k}\right) . \end{array}$$

Definition 4.3.3. *A* **regular map** *from X to X' is a map $\varphi : X \to X'$ such that for each $x \in X$ there exists a neighborhood $\mathcal{U}$ of x and an affine neighborhood $\mathcal{V}$ of $\varphi(x)$ for which $\varphi(\mathcal{U}) \subset \mathcal{V}$ and $\varphi : \mathcal{U} \to \mathcal{V}$ is given by m regular functions.*

Exercise 40
Check that the regularity condition at a point $x \in X$ is independent of the choice of the affine neighborhood containing $f(x)$.

Example. A **Veronese embedding of degree** d is a map $V : \mathbb{P}^n \to \mathbb{P}^N$ where $N = \binom{n+d}{d} - 1$ and $[z_0 : \ldots : z_n]$ is mapped to the point with homogeneous coordinates given by the various monomials $z_0^{\lambda_0} \ldots z_n^{\lambda_n}$, the exponents $\lambda_0, \ldots , \lambda_n$ being nonnegative integers such that $\lambda_0 + \ldots + \lambda_n = d$.

Exercise 41
Check that the set of all homogeneous polynomials of degree d in $n+1$ variables $z_0, \ldots, z_n$ forms a vector space of dimension $\binom{n+d}{d}$.

In particular, if $n = 1$ and $d = 2$, the Veronese embedding is simply

$$\begin{array}{rccc} V: & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^2 \\ & [z_0 : z_1] & \longmapsto & [z_0^2 : z_0 z_1 : z_1^2] \end{array} .$$

Exercise 42
Check that V is a regular map.

The Veronese embedding of degree d allows to translate the study of some problems concerning hypersurfaces of degree d in $\mathbb{P}^n$ into the case of hyperplanes in $\mathbb{P}^N$. ♢

Definition 4.3.4. *An* **isomorphism** *from X to X' is a regular map $\varphi : X \to X'$ which is invertible by a regular map. The symbol $\simeq$ indicates an isomorphism. The group of isomorphisms $X \to X$ is denoted by* $\mathrm{Isom}(X)$*. The projective varieties X and X' are* **isomorphic** *when there exists an isomorphism between them.*

4.4 Rational Maps on Projective Varieties

If $X \subset \mathbb{P}^n$ and $X' \subset \mathbb{P}^m$ are projective varieties, the symbol $X \dashrightarrow X'$ still denotes a map defined on some open subset of X.

Let $X \subset \mathbb{P}^n$ be an irreducible projective variety.

Definition 4.4.1. *A* **rational function** *on X is a function $\mathbb{P}^n \dashrightarrow \mathbb{C}$ whose restriction to each affine neighborhood is a rational function on $\mathbb{C}^n$ whose numerator and denominator have the same degree and whose denominator does not vanish identically on X. The ring of rational functions on X is denoted $\mathcal{O}_X$.*

Let $\mathbb{C}(X)$ denote the field of fractions of $\mathbb{C}[X]$, and let $\mathcal{I}_X$ be the ideal in $\mathcal{O}_X$ formed by the rational functions on $\mathbb{C}^n$ whose numerator vanishes identically on X.

Exercise 43
Show that the field $\mathbb{C}(X)$ is isomorphic to the quotient $\mathcal{O}_X/\mathcal{I}_X$.

Let $X \subset \mathbb{P}^n$ and $X' \subset \mathbb{P}^m$ be projective varieties.

Definition 4.4.2. *A* **rational map from** X **to** $\mathbb{P}^m$ *is a map $X \dashrightarrow \mathbb{P}^m$ which is given in homogeneous coordinates for $\mathbb{P}^m$ by $m+1$ rational functions on X. A* **rational map from** X **to** X' *is the restriction to X' of a rational map $\varphi : X \dashrightarrow \mathbb{P}^m$ such that there is an open set $\mathcal{U} \subset X$ where φ is regular and $\varphi(\mathcal{U}) \subset X'$.*

Definition 4.4.3. *A* **birational equivalence** *from X to X' is a rational map $X \dashrightarrow X'$ which is invertible by a rational map. The projective varieties X and X' are* **birationally equivalent** *when there exists a birational equivalence between them.*

Exercise 44
Show that X and X' are birationally equivalent if and only if the associated fields of rational functions, $\mathbb{C}(X)$ and $\mathbb{C}(X')$, are isomorphic.

4.5 Quasiprojective Varieties

The notions before make sense in the broader class of *quasiprojective varieties*, which encompasses both affine and projective varieties.

Definition 4.5.1. *A* **quasiprojective variety** *is a nonempty open subset of a projective variety.*

Zariski topology, regular function, regular map, isomorphism, rational function, rational map and birational equivalence are defined for quasiprojective varieties analogously to projective varieties. For instance, a **(Zariski) closed set** of a quasiprojective variety is the intersection of the variety with a closed subset of projective space. From now on, the term *variety* (without specifying affine or projective) refers to a quasiprojective variety.

It is a fact that two irreducible varieties X and X' are birationally equivalent if and only if they contain isomorphic open subsets $\mathcal{U} \subset X$ and $\mathcal{U}' \subset X'$ [44].

Example. The tautological line bundle L defined in Section 3.4 is a quasiprojective variety (yet not affine, nor projective). In fact, the inclusion of $\mathbb{C}^n$ in $\mathbb{P}^n$ as the open set of points $[z_0 : \ldots : z_n]$ with $z_n \neq 0$ induces an inclusion of the blow-up of $\mathbb{C}^n$ at the origin as an open subset of the projective variety $B(\mathbb{P}^n, [0 : \ldots : 0 : 1])$; cf. Section 4.3.

The blow-down map $\beta : L \to \mathbb{C}^n$ is a birational equivalence in the category of quasiprojective varieties. $\diamondsuit$

Blowing-up at a point is a local operation which extends to any quasiprojective variety modeled on the blow-up of $\mathbb{C}^n$ at the origin or of $\mathbb{P}^n$ at $[0 : \ldots : 0 : 1]$.

Exercise 45
Check that $B(\mathbb{P}^n, [0 : \ldots : 0 : 1]) \simeq \mathbb{P}(L \oplus \mathbb{C})$, where $\mathbb{P}(L \oplus \mathbb{C})$ was discussed in Section 3.4, and that the blow-down map $\beta : \mathbb{P}(L \oplus \mathbb{C}) \to \mathbb{P}^n$ is a birational equivalence in the category of projective varieties.

An irreducible projective variety X is **normal** if every point has a normal affine neighborhood.

A **normalization** of an irreducible variety X is an irreducible normal variety $\widetilde{X}$, together with a regular map $\nu : \widetilde{X} \to X$ which is a *finite* birational equivalence. A map $\varphi : X \to X'$ between varieties is **finite**, if any point $x \in X'$ has an affine neighborhood $\mathcal{V}$ such that the preimage $\mathcal{U} := \varphi^{-1}(\mathcal{V})$ is affine and the restriction $\varphi : \mathcal{U} \to \mathcal{V}$ is a finite map, that is every point has a finite number of preimages.

Any variety is a finite union of irreducible varieties [44]. If $X = \cup_i X_i$ presents X as a finite union of irreducible closed subsets and $X_i \not\subset X_j$ for all $i \neq j$, then the X_i's are called **irreducible components** of X.

One can define a normalization of an arbitrary variety X as a disjoint union of normalizations for each of the irreducible components of X.

4.6 Toric Varieties

The n-complex-dimensional **algebraic torus** $(\mathbb{C}^*)^n$ is a $2n$-dimensional Lie group under multiplication of complex numbers. The **weight lattice** of $(\mathbb{C}^*)^n$ is the lattice $\mathbb{Z}^n$. A **character** (or a *Laurent monomial*) of $(\mathbb{C}^*)^n$ is a group homomorphism $(\mathbb{C}^*)^n \to \mathbb{C}^*$.

There is a bijective correspondence between weights and characters of $(\mathbb{C}^*)^n$:

$$\begin{array}{lcl} \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n & \longleftrightarrow & \lambda : (\mathbb{C}^*)^n \to \mathbb{C}^* \\ & & w = (w_1, \dots, w_n) \mapsto w^\lambda := w_1^{\lambda_1} \cdot \ldots \cdot w_n^{\lambda_n} \ . \end{array}$$

An **action** of a torus $(\mathbb{C}^*)^n$ on a variety X is a group homomorphism $\psi : (\mathbb{C}^*)^n \to \mathrm{Isom}(X)$.

Definition 4.6.1. *A* **toric variety** *is an irreducible variety*[3] *X equipped with an action of an algebraic torus having an open dense orbit.*

Definition 4.6.2. *Two toric varieties are* **equivalent** *if there exists an equivariant isomorphism between them.*

Toric varieties are easy to construct, as the following examples illustrate.

Examples.

1. Let $A = \{\lambda^{(1)}, \dots, \lambda^{(k)}\}$ be a finite subset of $\mathbb{Z}^n$. Associated to A, there is an action of $(\mathbb{C}^*)^n$ on the projective space $\mathbb{P}^{k-1}$ defined by:

$$w \cdot [z_1 : \ldots : z_k] = [w^{\lambda^{(1)}} z_1 : \ldots : w^{\lambda^{(k)}} z_k] \ , \qquad \text{for } w \in (\mathbb{C}^*)^n \ .$$

Let X_A be the closure of the $(\mathbb{C}^*)^n$-orbit through $[1 : \ldots : 1]$. Then X_A is a toric variety.

[3] Toric varieties are usually required to be normal and normality is always the case for smooth varieties, which interest us mostly. However, our discussion does not require normality and is shortened without this assumption.

2. Let $A = \{\lambda^{(1)}, \ldots, \lambda^{(k)}\}$ be a finite subset of $\mathbb{Z}^n$. The action of $(\mathbb{C}^*)^n$ on the vector space $\mathbb{C}^k$ associated to A is defined by:
$$w \cdot (z_1, \ldots, z_k) = (w^{\lambda^{(1)}} z_1, \ldots, w^{\lambda^{(k)}} z_k) \ , \qquad \text{for } w \in (\mathbb{C}^*)^n \ .$$
Let Y_A be the closure of the $(\mathbb{C}^*)^n$-orbit through $(1, \ldots, 1)$. Then Y_A is a toric variety.

3. Suppose that we have an action of a torus $(\mathbb{C}^*)^n$ on some variety V. Let $v \in V$. The closure of the $(\mathbb{C}^*)^n$-orbit through v is a toric variety.

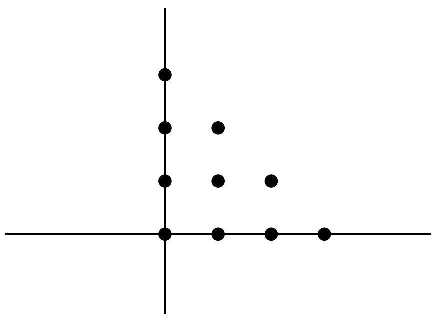
$\diamondsuit$

The important example X_A has the following particular instances.

Examples.

1. For a fixed positive integer d, let
$$\begin{aligned} A_1 &= \{\lambda = (\lambda_0, \ldots, \lambda_n) \in \mathbb{Z}^{n+1} \mid \lambda_i \geq 0 \text{ all } i \ , \lambda_0 + \ldots + \lambda_n = d\} \\ &= \{\lambda^{(0)}, \ldots, \lambda^{(N)}\} \end{aligned}$$
(which corresponds to the set of all Laurent monomials in n variables of degree d containing no negative powers). Then
$$X_{A_1} = \text{ closure of } \{[w^{\lambda^{(0)}} : \ldots : w^{\lambda^{(N)}}] \mid w \in (\mathbb{C}^*)^{n+1}\} = V(\mathbb{P}^n) \simeq \mathbb{P}^n$$
where $V : \mathbb{P}^n \to \mathbb{P}^N$ is the Veronese embedding of degree d defined in Section 4.3.

2. Similarly, for a fixed positive integer d, let
$$\begin{aligned} A_2 &= \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_i \geq 0 \text{ all } i \ , \lambda_1 + \ldots + \lambda_n \leq d\} \\ &\simeq \{\lambda = (\lambda_0, \ldots, \lambda_n) \in \mathbb{Z}^{n+1} \mid \lambda_i \geq 0 \text{ all } i \ , \lambda_0 + \ldots + \lambda_n = d\} \\ &= \{\lambda^{(0)}, \ldots, \lambda^{(N)}\} \ . \end{aligned}$$
As before $X_{A_2} = V(\mathbb{P}^n) \simeq \mathbb{P}^n$. For $n = 2$ and $d = 3$, the set A_2 is:



$\diamondsuit$

Chapter 5

Classification

Recall that a toric variety is an irreducible quasiprojective variety equipped with an action of an algebraic torus having an open dense orbit. In this lecture we begin by reviewing the language of spectra used for classifying affine toric varieties. Arbitrary normal toric varieties are classified by combinatorial objects called fans.

5.1 Spectra

Let A be a finitely generated $\mathbb{C}$-algebra without zero divisors. An ideal I in A is **prime** if

$$u, v \in A \text{ and } uv \in I \implies u \in I \text{ or } v \in I \ .$$

An ideal I in A is **maximal** if $I \neq A$ and the only proper ideal in A containing I is I itself.

Exercise 46
Regard A simply as a commutative ring with unity. Show that the ideal I is prime if and only if the quotient ring A/I is an integral domain (i.e., A/I has no zero divisors), and that the ideal I is maximal if and only if A/I is a field.

Exercise 47
Check that every maximal ideal is prime. Give an example of a prime ideal which is not maximal.

Exercise 48
Let I be an ideal in A, and let $p : A \to A/I$ be the surjective ring homomorphism given by taking an element to its coset in the quotient ring A/I. Check that there exists a bijective correspondence between the ideals J of A which contain I, and the ideals $\overline{J}$ of A/I, given by $J = p^{-1}(\overline{J})$.

The **spectrum** of the algebra A is the set

$$\mathrm{Spec}\ A := \{ \text{ prime ideals in } A \ \}$$

equipped with the **Zariski topology**, which declares to be **closed** a subset of Spec A consisting of all prime ideals containing some subset of A. The **maximal spectrum** of A is the set

$$\mathrm{Spec_m}\ A := \{ \text{ maximal ideals in } A \ \}$$

equipped with the **Zariski topology**, which declares to be **closed** a subset of $\mathrm{Spec_m}\ A$ consisting of all maximal ideals containing some subset of A.

Examples.

1. Let $A = \mathbb{C}[z_1, \ldots, z_n]$ and let $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$. Associated to the point x there is a maximal ideal $I_x \subset A$ consisting of all polynomials which vanish at x, that is, I_x is the ideal generated by the monomials $z_1 - x_1, \ldots, z_n - x_n$:

   $$I_x := \langle z_1 - x_1, \ldots, z_n - x_n \rangle \ .$$

   By Hilbert's Nullstellensatz (see, for instance, [5] or [20]), any maximal ideal of $\mathbb{C}[z_1, \ldots, z_n]$ is of the form I_x for some $x \in \mathbb{C}^n$. Moreover, the correspondence

   $$\mathbb{C}^n \simeq \mathrm{Spec_m}\ \underbrace{\mathbb{C}[z_1, \ldots, z_n]}_{\mathbb{C}[\mathbb{C}^n]}$$

   is a homeomorphism for the Zariski topology, where we stress that the polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$ is the ring of regular functions on $\mathbb{C}^n$.

   Exercise 49
   Show that there exists a bijective correspondence between Spec $\mathbb{C}[z_1, \ldots, z_n]$ and the set of irreducible subvarieties in $\mathbb{C}^n$.

   Hint: Any prime ideal in $\mathbb{C}[z_1, \ldots, z_n]$ is finitely generated.

2. Let $A = \mathbb{C}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]$ and let $x = (x_1, \ldots, x_n) \in (\mathbb{C}^*)^n$. Associated to the point x there is the maximal ideal in A:

   $$I_x := \langle z_1 - x_1, z_1^{-1} - x_1^{-1}, \ldots, z_n - x_n, z_n^{-1} - x_n^{-1} \rangle \ .$$

   By observing that the ideal $\langle z_1 - x_1, z_1^{-1} - y_1, \ldots, \rangle$ is the full algebra A when $y_1 \neq x_1^{-1}$ since it contains $(z_1 - x_1)z_1^{-1} + x_1(z_1^{-1} - y_1) \in \mathbb{C}^*$, Hilbert's Nullstellensatz implies that any maximal ideal of $\mathbb{C}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]$ is of the form I_x. Moreover, the correspondence

   $$(\mathbb{C}^*)^n \simeq \mathrm{Spec_m}\ \underbrace{\mathbb{C}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]}_{\mathbb{C}[(\mathbb{C}^*)^n]}$$

   is a homeomorphism for the Zariski topology, where $\mathbb{C}[(\mathbb{C}^*)^n]$ is the ring of regular functions on $(\mathbb{C}^*)^n$; cf. Section 4.1.

Exercise 50
Show that there exists a bijective correspondence between Spec $\mathbb{C}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]$ and the set of irreducible subvarieties in $(\mathbb{C}^*)^n$.

$\diamondsuit$

Let X be an affine variety in $\mathbb{C}^n$ defined by polynomials $p_1, \ldots, p_r$ from $\mathbb{C}[z_1, \ldots, z_n]$. Let $I = \langle p_1, \ldots, p_r \rangle$ be the ideal generated by those polynomials. Then

$$\begin{aligned}
X &= \{ \ x \in \mathbb{C}^n \mid p(x) = 0 \ , \ \forall p \in I \ \} \\
&= \{ \ x \in \mathbb{C}^n \mid I \subseteq I_x \ \} \\
&\simeq \{ \ \text{ideals } I_x \subset \mathbb{C}[z_1, \ldots, z_n] \mid I \subseteq I_x \ \} \\
&= \{ \ \text{maximal ideals } J \subset \mathbb{C}[z_1, \ldots, z_n] \mid I \subseteq J \ \} \\
&= \{ \ \text{maximal ideals in } \mathbb{C}[z_1, \ldots, z_n]/I \ \} \\
&= \{ \ \text{maximal ideals in } \mathbb{C}[X] \ \} \\
&= \operatorname{Spec}_{\mathrm{m}} \mathbb{C}[X] \ .
\end{aligned}$$

The correspondence $X \simeq \operatorname{Spec}_{\mathrm{m}} \mathbb{C}[X]$ is a homeomorphism for the Zariski topology.

More generally, if A is a finitely generated $\mathbb{C}$-algebra without zero divisors, the set $X_A := \operatorname{Spec}_{\mathrm{m}} A$ is called an **abstract affine variety**. While maximal ideals in A play the role of points in X_A, arbitrary prime ideals are thought of as irreducible subvarieties, by analogy with the ring of regular functions on an affine variety. The **dimension** of X_A is defined to be

$$\dim X_A := \sup_{n \in \mathbb{Z}} \{ \exists \text{ chain } I_0 \subset I_1 \subset \ldots \subset I_n \text{ of distinct prime ideals} \} \ .$$

When $A = \mathbb{C}[X]$ is the ring of regular functions on an irreducible affine variety X, the dimension $\dim X_A$ coincides with the (complex) dimension $\dim X$ since both are equal to

$$\sup_{n \in \mathbb{Z}} \{ \exists \text{ chain } X_0 \subset X_1 \subset \ldots \subset X_n = X \text{ of distinct irreducible subvarieties} \} \ .$$

5.2 Toric Varieties Associated to Semigroups

Let S be a commutative semigroup. Its **semigroup algebra** $\mathbb{C}[S]$ is the $\mathbb{C}$-algebra generated as a complex vector space by the symbols z^σ with $\sigma \in S$ and multiplication defined by the rule

$$z^\sigma \cdot z^{\sigma'} = z^{\sigma + \sigma'} \ .$$

In particular, generators σ_i for S as a semigroup yield generators z^{σ_i} for $\mathbb{C}[S]$ as a $\mathbb{C}$-algebra.

Examples.

1. If $S = (\mathbb{Z}_0^+)^n$, then $\mathbb{C}[S] = \mathbb{C}[z_1, \ldots, z_n]$ is the algebra of polynomials in n variables.

2. If $S = \mathbb{Z}^n$, then $\mathbb{C}[S] = \mathbb{C}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]$ is the algebra of Laurent polynomials in n variables.

$\diamondsuit$

Notice that in the previous two examples, the maximal spectrum of the semigroup algebras are toric varieties:

$$\mathrm{Spec_m}\ \mathbb{C}[(\mathbb{Z}_0^+)^n] \simeq \mathbb{C}^n \quad \text{and} \quad \mathrm{Spec_m}\ \mathbb{C}[\mathbb{Z}^n] \simeq (\mathbb{C}^*)^n \ .$$

In fact, we have the following general result:

Proposition 5.2.1. *Let $S \subseteq \mathbb{Z}^n$ be a finitely generated semigroup. Then the maximal spectrum* $\mathrm{Spec_m}\ \mathbb{C}[S]$ *is an affine toric variety.*

Proof. By shrinking the lattice $\mathbb{Z}^n$ if necessary, we may assume that S generates $\mathbb{Z}^n$ as an abelian group, in which case $\mathrm{Spec_m}\ \mathbb{C}[S]$ has dimension n. The inclusion of semigroups $S \subseteq \mathbb{Z}^n$ and hence of semigroup algebras $\mathbb{C}[S] \subset \mathbb{C}[\mathbb{Z}^n]$ gives an embedding of the torus $(\mathbb{C}^*)^n$:

$$(\mathbb{C}^*)^n = \mathrm{Spec_m}\ \mathbb{C}[\mathbb{Z}^n] \hookrightarrow \mathrm{Spec_m}\ \mathbb{C}[S] \ .$$

Let $\mathcal{O}$ be the image of this embedding. The torus $(\mathbb{C}^*)^n$ acts on $\mathbb{C}[S]$ by

$$w \cdot z^\sigma := w^\sigma z^\sigma \ , \quad \text{for } w \in (\mathbb{C}^*)^n \text{ and } \sigma \in S \ .$$

This action induces an action of $(\mathbb{C}^*)^n$ on $\mathrm{Spec_m}\ \mathbb{C}[S]$. By considering the dimension, the set $\mathcal{O} \simeq (\mathbb{C}^*)^n$ is an open orbit for this action. We conclude that its closure $\overline{\mathcal{O}}$ must be the full $\mathrm{Spec_m}\ \mathbb{C}[S]$ and hence this is a toric variety. $\square$

Example. The complex curve in $\mathbb{C}^2$ with equation $y^k = x^{k+1}$ $(k = 1, 2, 3, \ldots)$ is an affine toric variety with $\mathbb{C}^*$-action given by

$$t \cdot (x, y) = (t^k x, t^{k+1} y) \ .$$

It may be obtained as $\mathrm{Spec_m}\ \mathbb{C}[S]$ for the semigroup $S = \mathbb{Z}_0^+ \setminus \{1, 2, \ldots, k-1\}$ generated by $\{k, k+1, \ldots, 2k-1\}$. $\diamondsuit$

Exercise 51
Show that the variety in the previous example is not normal for $k > 1$, and that its normalization is the affine line X.

5.3 Classification of Affine Toric Varieties

Theorem 5.3.1. (classification of affine toric varieties) *Any affine toric variety is equivalent to one of the form* $\mathrm{Spec_m}\ \mathbb{C}[S]$ *for some finitely generated semigroup* $S \subset \mathbb{Z}^n$ *(*$n \geq 0$*).*

Proof. Let X be an affine toric variety for the torus $(\mathbb{C}^*)^m$. Let $\mathcal{O}$ be the open orbit for the $(\mathbb{C}^*)^m$-action on X. Then $\mathcal{O}$ may be identified with the quotient of $(\mathbb{C}^*)^m$ by the stabilizer of some point in $\mathcal{O}$. Since this quotient is itself a (possibly smaller dimensional) torus, we can regard $\mathcal{O}$ itself as a torus $(\mathbb{C}^*)^n$ acting on X. By irreducibility of X, we have an embedding $\mathbb{C}[X] \subset \mathbb{C}[\mathcal{O}] = \mathbb{C}[\mathbb{Z}^n]$. The subring $\mathbb{C}[X] \subset \mathbb{C}[\mathcal{O}]$ is $(\mathbb{C}^*)^n$-invariant with respect to the induced actions of $(\mathbb{C}^*)^n$ on $\mathbb{C}[X]$ and on $\mathbb{C}[(\mathbb{C}^*)^n]$. As a representation of an algebraic torus, the space $\mathbb{C}[X]$ decomposes into one-dimensional weight spaces. The weight spaces are generated by monomials as $\mathbb{C}$-algebras, hence the vector space $\mathbb{C}[X]$ itself is generated by monomials, i.e., it is a semigroup algebra. □

Example. Recall the construction of the affine toric variety Y_A from a finite set $A = \{\lambda^{(1)}, \ldots, \lambda^{(k)}\} \subset \mathbb{Z}^n$, described in Section 4.6. By the previous theorem, we must have $Y_A \simeq \mathrm{Spec_m}\ \mathbb{C}[S]$ for some finitely generated semigroup $S \subset \mathbb{Z}^n$. It is not hard to see that S is the semigroup generated by A. In fact, the ring $\mathbb{C}[Y_A]$ of regular functions on Y_A is generated by the restrictions to Y_A of the coordinate functions on $\mathbb{C}^n$. Since Y_A is the closure of

$$\{(z^{\lambda^{(1)}}, \ldots, z^{\lambda^{(k)}}) \mid z \in (\mathbb{C}^*)^n\} ,$$

the ring $\mathbb{C}[Y_A]$ is generated by the monomials

$$z^{\lambda^{(1)}}, \ldots, z^{\lambda^{(k)}} ,$$

i.e., is the semigroup algebra of the semigroup in $\mathbb{Z}^n$ generated by A. ♢

Remark. The only *smooth* affine toric varieties are products of the form $(\mathbb{C}^*)^p \times \mathbb{C}^q$. This follows from the classification of affine toric varieties (Theorem 5.3.1), the classification of normal toric varieties (Theorem 5.6.1) and the study of conditions for smoothness (Exercise 53). See the remark at the end of Section 6.4. ♢

5.4 Fans

Definition 5.4.1. *A* **(convex polyhedral) cone** *in* $\mathbb{R}^n$ *is a set of the form*

$$C = \{a_1 v_1 + \ldots + a_r v_r \in \mathbb{R}^n \mid a_1, \ldots, a_r \geq 0\}$$

for some finite set of vectors $v_1, \ldots, v_r \in \mathbb{R}^n$*, then called the* **generators** *of the cone* C*. The cone* C *is* **rational** *if it admits a set of generators in* $\mathbb{Z}^n$*. The cone* C *is* **smooth** *if it admits a set of generators which is part of some* $\mathbb{Z}$*-basis of* $\mathbb{Z}^n$*.*

The **dimension** of a cone C is the dimension of the smallest $\mathbb{R}$-subspace containing C (which is the vector space $C + (-C)$).

Definition 5.4.2. *The* **dual** *of a cone* $C \subset \mathbb{R}^n$ *is*

$$C^* := \{f \in (\mathbb{R}^n)^* \mid f(x) \geq 0 \; \forall x \in C\} \; .$$

Farkas' theorem states that the dual of a rational cone is a rational cone [21]. From the theory of convex sets [20], it follows that $(C^*)^* = C$.

A **supporting hyperplane** for a cone $C \subset \mathbb{R}^n$ is a hyperplane of the form

$$H_f := \{x \in \mathbb{R}^n \mid f(x) = 0\} \quad \text{for some } f \in C^* \setminus \{0\} \; .$$

A **face** of a cone $C \subset \mathbb{R}^n$ is either C itself (a nonproper face) or the intersection of C with any supporting hyperplane (proper faces). A face of a cone is itself a cone; indeed the face $C \cap H_f$ (with $f \in C^*$) is generated by those vectors v_i in a set of generators for C such that $f(v_i) = 0$.

Exercise 52
Show that a cone C has only finitely many faces and that any intersection of faces is also a face.

Hint: The face $C \cap H_f$ is generated by those vectors v_i in a generating set for C such that $f(v_i) = 0$.

If 0 is a face of C, then C is called **strongly convex**; this is the case precisely when C contains no one-dimensional $\mathbb{R}$-subspaces, that is, when $C \cap (-C) = \{0\}$. If C is strongly convex, then its dual is n-dimensional (i.e., $C^* + (-C^*) = (\mathbb{R}^n)^*$), regardless of the dimension of C.

Let C be a rational cone in $\mathbb{R}^n$, and let C^* be its dual, also rational.

Lemma 5.4.3. *The intersection* $C^* \cap (\mathbb{Z}^n)^*$ *is a finitely generated semigroup.*

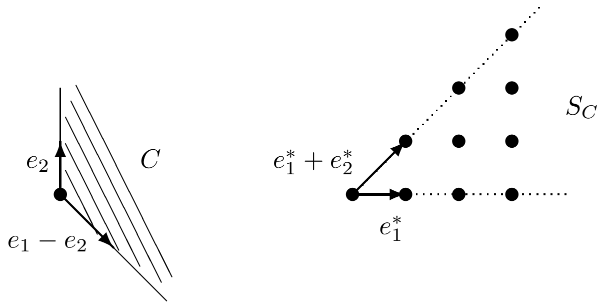
Proof. Let $v_1, \ldots, v_r \in (\mathbb{Z}^n)^*$ be generators of C^* and let $K = \{\sum t_i v_i \mid 0 \leq t_i \leq 1\}$. The intersection $K \cap (\mathbb{Z}^n)^*$ is finite since K is compact. It suffices to show that $K \cap (\mathbb{Z}^n)^*$ generates S_C. For $v \in S_C$, write $v = \sum r_i v_i$ where $r_i \geq 0$, so $r_i = m_i + t_i$ with m_i a nonnegative integer and $0 \leq t_i \leq 1$. Then $v = \sum m_i v_i + \sum t_i v_i$ with each v_i and $\sum t_i v_i$ in $K \cap (\mathbb{Z}^n)^*$. □

The finitely generated semigroup $C^* \cap (\mathbb{Z}^n)^*$ is denoted S_C.

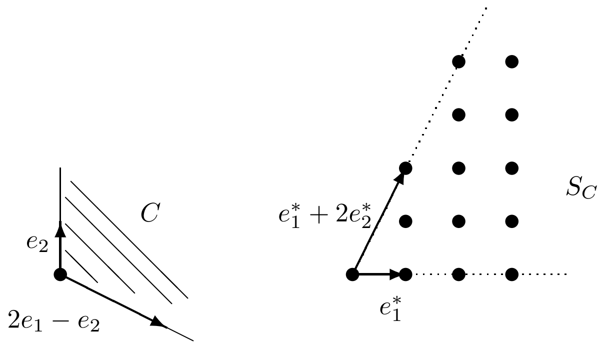
Examples.

1. For the cone C given by the first octant in $\mathbb{Z}^n$, the semigroup S_C consists of elements in $(\mathbb{Z}^n)^*$ with all coordinates nonnegative, and is generated by $e_1^*, \ldots, e_n^*$, where $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1) \in \mathbb{Z}^n$.

2. For the trivial strongly convex cone $C = \{0\}$ in $\mathbb{Z}^n$, the semigroup $S_C = (\mathbb{Z}^n)^*$ is generated by $e_1^*, -e_1^*, \ldots, e_n^*, -e_n^*$.
3. For the cone $C \subset \mathbb{Z}^2$ generated by e_2 and $e_1 - e_2$, the semigroup S_C is generated by e_1^* and $e_1^* + e_2^*$.



4. For the cone $C \subset \mathbb{Z}^2$ generated by e_2 and $2e_1 - e_2$, the semigroup S_C is generated by e_1^*, $e_1^* + e_2^*$ and $e_1^* + 2e_2^*$.



$\diamondsuit$

Corollary 5.4.4. *For a rational cone $C \subset \mathbb{R}^n$, the affine variety* $\operatorname{Spec}_{\mathrm{m}} \mathbb{C}[S_C]$ *is a toric variety.*

Proof. This follows immediately from Proposition 5.2.1 and Lemma 5.4.3. $\square$

Examples.

1. For the cone C given by the first quadrant in $\mathbb{Z}^n$, the associated semigroup algebra is
$$\mathbb{C}[S_C] = \mathbb{C}[z_1, \ldots, z_n]$$

and the corresponding toric variety is

$$\mathrm{Spec}_{\mathrm{m}}\ \mathbb{C}[S_C] \simeq \mathbb{C}^n\ .$$

2. For the trivial cone $C = \{0\}$ in $\mathbb{Z}^n$, the associated semigroup algebra is

$$\mathbb{C}[S_C] = \mathbb{C}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]$$

and the corresponding toric variety is the torus

$$\mathrm{Spec}_{\mathrm{m}}\ \mathbb{C}[S_C] \simeq (\mathbb{C}^*)^n\ .$$

3. For the third cone in the previous list of examples, the associated semigroup algebra is

$$\mathbb{C}[S_C] = \mathbb{C}[z_1, z_1 z_2] \simeq \mathbb{C}[w_1, w_2]$$

and the corresponding toric variety is

$$\mathrm{Spec}_{\mathrm{m}}\ \mathbb{C}[S_C] \simeq \mathbb{C}^2\ .$$

4. For the fourth cone in the previous list of examples, the associated semigroup algebra is

$$\mathbb{C}[S_C] = \mathbb{C}[z_1, z_1 z_2, z_1 z_2^2] \simeq \mathbb{C}[w_1, w_2, w_3]/\langle w_1 w_3 = w_2^2\rangle$$

and the corresponding toric variety is the quadratic cone

$$\mathrm{Spec}_{\mathrm{m}}\ \mathbb{C}[S_C] \simeq \{(w_1, w_2, w_3) \in \mathbb{C}^3 \mid w_1 w_3 = w_2^2\}$$

which has an orbifold $\mathbb{Z}_2$ singularity at the origin. Note that, for the cone $C \subset \mathbb{Z}^2$ generated by e_2 and $e_1 - 2e_2$, the associated semigroup algebra is

$$\mathbb{C}[S_C] = \mathbb{C}[z_1, z_1^2 z_2] \simeq \mathbb{C}[w_1, w_2]\ ,$$

which corresponds to a smooth toric variety.

$\diamondsuit$

If we have an inclusion of rational cones $C \subset C'$, then $(C')^* \subset C^*$, hence $\mathbb{C}[S_{C'}]$ is a subalgebra of $\mathbb{C}[S_C]$. It follows that we get a map

$$\mathrm{Spec}_{\mathrm{m}}\ \mathbb{C}[S_C] \ \longrightarrow\ \mathrm{Spec}_{\mathrm{m}}\ \mathbb{C}[S_{C'}]\ ;$$

to see this, first notice that, since nontrivial $\mathbb{C}$-algebra homomorphisms $\mathbb{C}[S_C] \to \mathbb{C}$ are uniquely determined by their kernels which are exactly the maximal ideals of $\mathbb{C}[S_C]$, we obtain a bijection

$$\mathrm{Spec}_{\mathrm{m}}\ \mathbb{C}[S_C] \ \longleftrightarrow\ \mathrm{Hom}_{\,\mathbb{C}\text{-alg}}(\mathbb{C}[S_C], \mathbb{C}) \setminus \{0\}\ .$$

Thus, by restricting homomorphisms from $\mathbb{C}[S_C]$ to $\mathbb{C}[S_{C'}]$, we get the asserted map.

Lemma 5.4.5. *Let C and C' be rational cones. If C is a face of C', then the induced map* $\mathrm{Spec_m}\ \mathbb{C}[S_C] \to \mathrm{Spec_m}\ \mathbb{C}[S_{C'}]$ *is an open injection for the Zariski topology.*

In other words, if C is a face of C', then $\mathrm{Spec_m}\ \mathbb{C}[S_C]$ is an open subset of $\mathrm{Spec_m}\ \mathbb{C}[S_{C'}]$.[1]

Proof. If C is a face of C', then (see, for instance, [21]) there is $v \in S_{C'}$ such that

$$S_C = S_{C'} + \mathbb{Z}_0^+(-v) \ ,$$

thus each element of $\mathbb{C}[S_C]$ may be written in the form $z^{\sigma - nv}$ for some $\sigma \in S_{C'}$ and $n \in \mathbb{Z}_0^+$. The map $\mathrm{Spec_m}\ \mathbb{C}[S_C] \to \mathrm{Spec_m}\ \mathbb{C}[S_{C'}]$ is injective, since if two $\mathbb{C}$-algebra homomorphisms $\mathbb{C}[S_C] \to \mathbb{C}$ coincide on $\mathbb{C}[S_{C'}]$, then they also coincide on elements z^{-nv}. The map $\mathrm{Spec_m}\ \mathbb{C}[S_C] \to \mathrm{Spec_m}\ \mathbb{C}[S_{C'}]$ misses exactly the maximal ideals in $\mathbb{C}[S_{C'}]$ containing the set $\{z^v\}$, since any $\mathbb{C}$-algebra homomorphism $\mathbb{C}[S_{C'}] \to \mathbb{C}$ which does not vanish on z^v extends to a nontrivial homomorphism $h : \mathbb{C}[S_C] \to \mathbb{C}$ where $h(z^{-v}) = h(z^v)^{-1}$, and if a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[S_{C'}] \to \mathbb{C}$ vanishes on z^v, then any extension $h : \mathbb{C}[S_C] \to \mathbb{C}$ must vanish identically since $h(1) = h(z^v z^{-v}) = h(z^v)h(z^{-v}) = 0$. $\square$

Definition 5.4.6. *A* **fan** *in $\mathbb{R}^n$ is a (nonempty) finite collection $\mathcal{F}$ of strongly convex rational cones such that*

- *every face of every cone $C \in \mathcal{F}$ belongs to $\mathcal{F}$,*
- *the intersection of any two cones from $\mathcal{F}$ is a face of both of them.*

The fan $\mathcal{F}$ is **smooth** *if all of its cones are smooth. The* **support** *of $\mathcal{F}$ is the union $|\mathcal{F}|$ of all cones from $\mathcal{F}$. The fan $\mathcal{F}$ is* **complete** *if $|\mathcal{F}|$ is the whole space.*

5.5 Toric Varieties Associated to Fans

Definition 5.5.1. *The* **toric variety $X_{\mathcal{F}}$ associated to a fan $\mathcal{F}$** *in $\mathbb{R}^n$ is the result of gluing the affine toric varieties $X_C := \mathrm{Spec_m}\ \mathbb{C}[S_C]$ (for all $C \in \mathcal{F}$) by identifying X_C with the correponding Zariski open subset in $X_{C'}$ whenever C is a face of C'.*

Each affine chart X_C has a natural torus action defined as in the proof of Proposition 5.2.1. Those actions are compatible under the identifications dictated by the face relations; hence there is a well-defined torus action on the variety $X_{\mathcal{F}}$. Moreover, $X_{\mathcal{F}}$ contains indeed an open dense orbit of $(\mathbb{C}^*)^n$ as the open set corresponding to the zero cone in $\mathcal{F}$: by strong convexity the zero cone is a face of every other cone, thus producing an open subset of each other affine piece, and

[1] This functorial property partly justifies working with cones and not their duals right away.

the dual of the zero cone is the full set $(\mathbb{R}^n)^*$, so the corresponding algebra is $\mathbb{C}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]$ whose maximal spectrum is $(\mathbb{C}^*)^n$.

The variety $X_\mathcal{F}$ is normal since it is glued out of normal affine varieties and normality is a local property.

Exercise 53
Show that:

(a) The variety $X_\mathcal{F}$ is compact if and only if the fan $\mathcal{F}$ is complete.

(b) The variety $X_\mathcal{F}$ is smooth if and only if every cone from $\mathcal{F}$ is smooth.

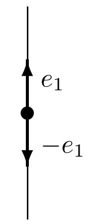
As a consequence of the first part of the previous exercise, if $\mathcal{F}$ is a complete fan in $\mathbb{R}^n$, then $X_\mathcal{F}$ is a compactification of the torus $(\mathbb{C}^*)^n$.

Examples.

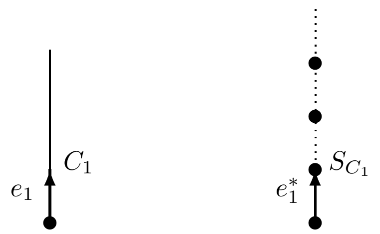
1. Consider the fan $\mathcal{F}$ consisting of the three cones

$$C = \{0\} \ , \quad C_0 = \mathbb{Z}_0^+(-e_1) \quad \text{and} \quad C_1 = \mathbb{Z}_0^+(e_1)$$

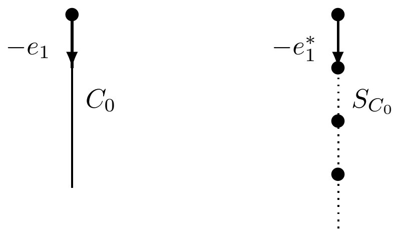
depicted below.



Each 1-dimensional cone represents the affine variety $\mathbb{C}$:

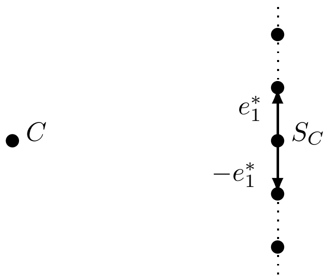


$$\mathbb{C}[S_{C_1}] = \mathbb{C}[z]$$



$\mathbb{C}[S_{C_0}] = \mathbb{C}[z^{-1}]$

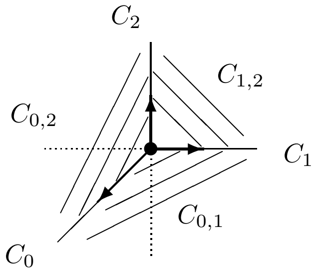
The gluing of these 1-dimensional charts is prescribed by the 0-dimensional cone C representing $\mathbb{C}^*$:



$\mathbb{C}[S_C] = \mathbb{C}[z, z^{-1}]$

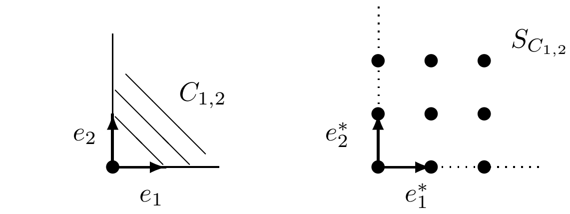
In X_{C_1}, the subset X_C corresponds to $\mathbb{C}_z^* := \{z \in \mathbb{C} \mid z \neq 0\}$, whereas in X_{C_0}, the subset X_C corresponds to $\mathbb{C}_{z^{-1}}^* := \{z^{-1} \in \mathbb{C} \mid z^{-1} \neq 0\}$. We can glue X_{C_1} to X_{C_0} along X_C by using the gluing map $z \mapsto z^{-1}$, thus producing $X_{\mathcal{F}} = \mathbb{P}^1$.

2. Consider the fan $\mathcal{F}$ consisting of seven cones (three 2-dimensional cones, three 1-dimensional cones and one 0-dimenisonal cone $C = \{0\}$), as sketched below. The shaded areas represent 2-dimensional cones.[2]



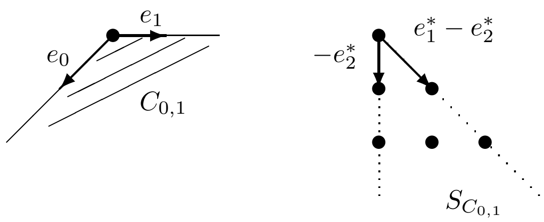
We will check that the toric variety $X_{\mathcal{F}}$ is $\mathbb{P}^2$. In fact, each 2-dimensional cone corresponds to an affine chart $\mathbb{C}^2$:

[2] If instead of the cones we considered their duals, the drawing would be messy with overlappings, hence the reason why we stick to this side of duality.



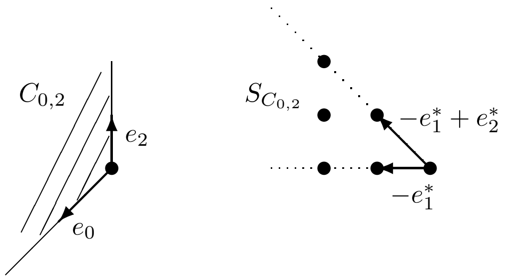
$$\mathbb{C}[S_{C_{1,2}}] = \mathbb{C}[z_1, z_2]$$

$$(z_1, z_2) = (\tfrac{w_1}{w_0}, \tfrac{w_2}{w_0})$$



$$\mathbb{C}[S_{C_{0,1}}] = \mathbb{C}[z_2^{-1}, z_1 z_2^{-1}]$$

$$(z_2^{-1}, z_1 z_2^{-1}) = (\tfrac{w_0}{w_2}, \tfrac{w_1}{w_2})$$



$$\mathbb{C}[S_{C_{0,2}}] = \mathbb{C}[z_1^{-1}, z_1^{-1} z_2]$$

$$(z_1^{-1}, z_1^{-1} z_2) = (\tfrac{w_0}{w_1}, \tfrac{w_2}{w_1})$$

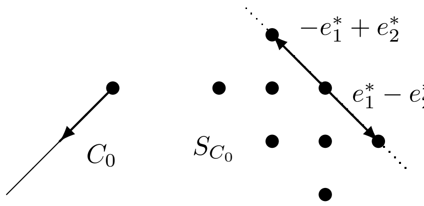
The expressions in terms of homogeneous coordinates $[w_0 : w_1 : w_2]$ on these three affine charts are written just to help keep track of the gluing maps below.[3]

The gluing of these affine charts along their intersections is prescribed by the 1-dimensional cones representing $\mathbb{C}^* \times \mathbb{C}$:

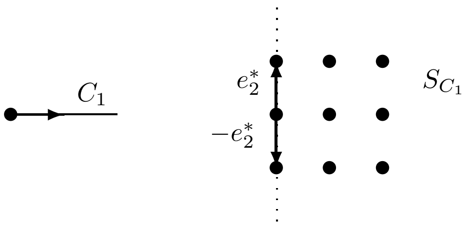
[3]The initial chosen identification

$$X_{C_{1,2}} \simeq \{[1 : z_1 : z_2] \mid (z_1, z_2) \in \mathbb{C}^2\}$$

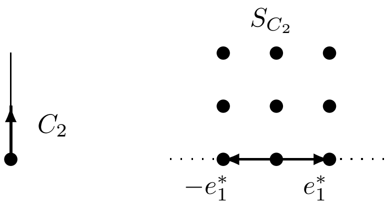
determines the other two.



$$\mathbb{C}[S_{C_0}] = \mathbb{C}[z_1^{-1}z_2, z_1z_2^{-1}, z_1^{-1}]$$



$$\mathbb{C}[S_{C_1}] = \mathbb{C}[z_1, z_2, z_2^{-1}]$$



$$\mathbb{C}[S_{C_2}] = \mathbb{C}[z_1, z_1^{-1}, z_2]$$

For instance, in $X_{C_{1,2}}$, the subset X_{C_1} is represented by $\mathbb{C}_{z_1} \times \mathbb{C}^*_{z_2}$ whereas in $X_{C_{0,1}}$, the subset X_{C_1} corresponds to $\mathbb{C}_{z_1 z_2^{-1}} \times \mathbb{C}^*_{z_2^{-1}}$. We can glue $X_{C_{1,2}}$ to $X_{C_{0,1}}$ along X_{C_1} by using the gluing map $(z_1, z_2) \mapsto (z_2^{-1}, z_1 z_2^{-1})$, to obtain $\mathbb{P}^2 \setminus \{[0:1:0]\}$.

3. Let $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$ be the standard $\mathbb{Z}$-basis of $\mathbb{Z}^n$, and let $e_0 := -e_1 - \ldots - e_n$. Let $C_{i_1,\ldots,i_k}$ be the cone in $\mathbb{R}^n$ generated by the vectors $e_{i_1}, \ldots e_{i_k}$. Then the set

$$\mathcal{F} := \{C_{i_1,\ldots,i_k} \mid k \leq n\ , 0 \leq i_j \leq n\}\}$$

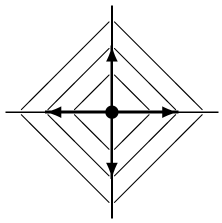
is a complete fan (where we include the trivial cone $C = \{0\}$.) In particular, for $n = 1, 2$ we get the fans encoding $\mathbb{P}^1$ and $\mathbb{P}^2$, respectively.

$\diamondsuit$

Exercise 54
Check that the toric variety associated to the fan $\mathcal{F}$ in the previous example is projective space:

$$X_{\mathcal{F}} \simeq \mathbb{P}^n \ .$$

Exercise 55
Find the toric variety corresponding to the fan depicted below.



5.6 Classification of Normal Toric Varieties

Theorem 5.6.1. (classification of normal toric varieties) *Any normal toric variety X is equivalent to a variety of the form $X_{\mathcal{F}}$ for some fan $\mathcal{F}$ in $\mathbb{R}^n$, where n is the dimension of the torus acting on X. This fan is determined uniquely up to a transformation from* $\mathrm{GL}(n;\mathbb{Z})$.

For a proof of Theorem 5.6.1, see for instance [41]. Thanks to this theorem, normal toric varieties are often defined in terms of fans.

Proposition 5.6.2. *Let $\mathcal{F}$ be a fan in $\mathbb{R}^n$. Then the variety $X_{\mathcal{F}}$ has finitely many orbits of the torus $(\mathbb{C}^*)^n$, and there is a natural bijection between the (nonempty) cones in $\mathcal{F}$ and the $(\mathbb{C}^*)^n$-orbits in $X_{\mathcal{F}}$,*

$$\left\{ \begin{array}{c} \textit{cones} \\ \textit{from } \mathcal{F} \end{array} \right\} \quad \longrightarrow \quad \left\{ \begin{array}{c} (\mathbb{C}^*)^n\textit{-orbits} \\ \textit{in } X_{\mathcal{F}} \end{array} \right\}$$

$$C \quad \longmapsto \quad \mathcal{O}_C$$

where the orbit $\mathcal{O}_C$ has dimension equal to the codimension of C. Moreover, for cones $C, C' \in \mathcal{F}$ we have

$$\mathcal{O}_{C'} \subset \overline{\mathcal{O}_C} \iff C \subset C' \ .$$

For a proof of Proposition 5.6.2, see again [41]. In the next lecture we state and prove the polytope analogue of this proposition.

Chapter 6

Moment Polytopes

We have seen that normal toric varieties are classified by fans. Most interesting for our purposes are those fans dual to polytopes: a fan associated to a polytope defines an equivariantly projective toric variety. Moreover, a polytope encodes other geometric information such as a symplectic form and an equivariant complex line bundle. We will relate the dual languages of polytopes and fans, and review the link to the symplectic approach.

6.1 Equivariantly Projective Toric Varieties

Let X be a toric variety for a torus $(\mathbb{C}^*)^n$. We say that X is **equivariantly projective** if there exists a $(\mathbb{C}^*)^n$-equivariant embedding $X \hookrightarrow \mathbb{P}^k$ for some k and some action of $(\mathbb{C}^*)^n$ on $\mathbb{P}^k$.

Let $A = \{\lambda^{(1)}, \ldots, \lambda^{(k)}\}$ be a finite subset of $\mathbb{Z}^n$. The first example of a toric variety in Section 4.6 is

$$X_A := \text{ closure of } \{[w^{\lambda^{(1)}} : \ldots : w^{\lambda^{(k)}}] \mid w \in (\mathbb{C}^*)^n\} \ ,$$

that is, the closure of the $(\mathbb{C}^*)^n$-orbit through $[1 : \ldots : 1]$ for the action on $\mathbb{P}^{k-1}$ defined by the weights $\lambda^{(i)}$, $i = 1, \ldots, k$. The variety X_A is clearly *equivariantly* embedded in $\mathbb{P}^{k-1}$, i.e., the action of the torus on X_A extends to the whole ambient $\mathbb{P}^{k-1}$. The following theorem shows that, conversely, any equivariantly projective toric variety is equivalent to one of type X_A for some finite set $A \subset \mathbb{Z}^n$.

Theorem 6.1.1. *Let X be a toric variety which is $(\mathbb{C}^*)^n$-equivariantly embedded in $\mathbb{P}^{\ell-1}$. Let Y be the minimal projective subspace in $\mathbb{P}^{\ell-1}$ containing X, and let $k-1$ be the (complex) dimension of Y. Then there exists a subset $A \subset \mathbb{Z}^n$ containing k elements and a $(\mathbb{C}^*)^n$-equivariant isomorphism $X_A \to X$ extending to an equivariant projective isomorphism $\mathbb{P}^{k-1} \to Y$.*

Proof. Any action of $(\mathbb{C}^*)^n$ on $\mathbb{P}^{\ell-1}$ by projective transformations can be lifted to a linear action on $\mathbb{C}^\ell$. Any linear action of $(\mathbb{C}^*)^n$ on $\mathbb{C}^\ell$ is diagonalizable, so in suitable coordinates it is given by a collection of weights $\lambda^{(i)} \in \mathbb{Z}^n$, $i = 1, \ldots, \ell$ such that $w \in (\mathbb{C}^*)^n$ acts as multiplication by

$$\begin{bmatrix} w^{\lambda^{(1)}} & & \\ & \ddots & \\ & & w^{\lambda^{(\ell)}} \end{bmatrix} .$$

Pick a point $z = [z_1 : \ldots : z_\ell] \in X$ lying on the open orbit of the torus. Let $A \subset \mathbb{Z}^n$ be the collection of those $\lambda^{(i)}$ for which $z_i \neq 0$. Then the minimal projective subspace of $\mathbb{P}^{\ell-1}$ containing X is

$$Y := \{[w_1 : \ldots : w_\ell] \in \mathbb{P}^{\ell-1} \mid z_i = 0 \Rightarrow w_i = 0\} .$$

The dimension of Y is $k - 1$ where $k = \#A$. We obtain a $(\mathbb{C}^*)^n$-equivariant isomorphism by collecting the nonzero coordinates:

$$\begin{array}{ccc} \mathbb{P}^{k-1} & \xrightarrow{\simeq} & Y \\ \bigcup & & \bigcup \\ X_A & \xrightarrow{\simeq} & X . \end{array}$$

□

Example. Not all projective toric varieties are equivariantly projective. For instance, the nodal[1] cubic curve $X \subset \mathbb{P}^2$ given by the equation

$$z_0 z_2^2 = z_1^3 - z_0 z_1^2$$

is a (not normal) toric variety as its smooth part,

$$X \setminus \{[1 : 0 : 0]\} \simeq \mathbb{C}^* ,$$

is an open orbit for $\mathbb{C}^*$. For a reason why X does not admit an equivariant projective embedding see [22, p.169]. ♢

6.2 Weight Polytopes

Consider a torus $(\mathbb{C}^*)^n$ acting linearly on a complex vector space V and consider the associated action on the projectivization $\mathbb{P}(V)$. Let v be a nonzero vector in V, and let $\overline{\mathcal{O}_v}$ be the closure in $\mathbb{P}(V)$ of the $(\mathbb{C}^*)^n$-orbit through $[v]$. By construction,

[1] The adjective *nodal* refers to having no singularities other than ordinary double points.

$\overline{\mathcal{O}_v}$ is a toric variety equivariantly embedded in $\mathbb{P}(V)$. By Theorem 6.1.1, the toric variety $\overline{\mathcal{O}_v}$ is equivalent to X_{A_v}, where A_v is the finite subset of $\mathbb{Z}^n$ prescribed as follows (adapting the proof of the previous theorem).

For a given a weight $\lambda \in \mathbb{Z}^n$ of $(\mathbb{C}^*)^n$, the **λ-weight space** of the $(\mathbb{C}^*)^n$-representation on V is the subspace

$$V_\lambda = \{v \in V \mid w \cdot v = w^\lambda v \ , \ \forall w \in (\mathbb{C}^*)^n\} \ .$$

The **weight space decomposition** of this representation is the isomorphism [11]

$$V \simeq \bigoplus_{\lambda \in \mathbb{Z}^n} V_\lambda \ ;$$

cf. Exercise 9. Given a vector $v \in V$, the **component of** v **of weight** λ is the component $v_\lambda \in V_\lambda$ of v in the weight space decomposition. Set

$$A_v := \{\lambda \in \mathbb{Z}^n \mid v_\lambda \neq 0\} \ .$$

Exercise 56
Show that the toric variety $\overline{\mathcal{O}_v}$ is equivalent to X_{A_v}.

Definition 6.2.1. *The* **weight polytope** P_v *of the vector* $v \in V \setminus \{0\}$ *is the convex hull in* $\mathbb{R}^n$ *of the set* A_v *described in the previous paragraph.*

Example. Given a finite set $A \subset \mathbb{Z}^n$, its convex hull P in $\mathbb{R}^n$ is the weight polytope of the vector $v = (1, \ldots, 1)$ for which the closure of the orbit through $[v]$ is X_A. $\diamondsuit$

6.3 Orbit Decomposition

indexorbit decomposition

Let $A = \{\lambda^{(1)}, \ldots, \lambda^{(k)}\}$ be a finite subset of $\mathbb{Z}^n$. The toric variety associated to A is (cf. Section 4.6)

$$X_A := \text{ closure of } \{[w^{\lambda^{(1)}} : \ldots : w^{\lambda^{(k)}}] \in \mathbb{P}^{k-1} \mid w \in (\mathbb{C}^*)^n\} \ .$$

Proposition 6.3.1. *Let P be the convex hull in $\mathbb{R}^n$ of the set A above. Then there is a bijection between the (nonempty) faces of the polytope P and the $(\mathbb{C}^*)^n$-orbits in X_A given by*

$$\left\{ \begin{array}{c} \textit{faces of} \\ \textit{polytope } P \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} (\mathbb{C}^*)^n\textit{-orbits} \\ \textit{in } X_A \end{array} \right\}$$

$$F \longmapsto X^0(F)$$

where

$$X^0(F) = \{[u] \in X_A \mid \forall \lambda : u_\lambda = 0 \Leftrightarrow \lambda \notin F\} \ .$$

The dimension of $X^0(F)$ is the dimension of F. Moreover, the closure $X(F)$ of $X^0(F)$ is equivalent to the toric variety $X_{A \cap F}$. If F and F' are two faces of P, then

$$X(F') \subset X(F) \iff F' \subset F \ .$$

Proof. For $u = (u_1, \ldots, u_k) \in \mathbb{C}^k$ such that $[u] := [u_1 : \ldots : u_k] \in X_A$, given $\lambda^{(i)} \in A$ we have that the $\lambda^{(i)}$-component of u is

$$u_{\lambda^{(i)}} = (0, \ldots, 0, u_i, 0, \ldots, 0) \ .$$

We first show that the correspondence is well-defined, i.e., that $X^0(F)$ is indeed an orbit. Of course, $X^0(F)$ is invariant as

$$[u] \in X^0(F) \Longrightarrow w \cdot [u] \in X^0(F) \ .$$

Besides, any two $[u], [u'] \in X^0(F)$ are related: to show that there exists indeed $w \in (\mathbb{C}^*)^n$ such that $u_\lambda = w^\lambda u_{\lambda'}$ for all $\lambda \in F$, notice that $u_\lambda = w_1^\lambda$ and $u_{\lambda'} = w_2^\lambda$ for some $w_1, w_2 \in (\mathbb{C}^*)^n$, hence just take $w = w_1 w_2^{-1}$.

To prove injectivity, we need to show that

$$F' \neq F \Longrightarrow X^0(F') \neq X^0(F) \ .$$

Without loss of generality, we assume that there exists $\lambda \in F'$ such that $\lambda \notin F$. Therefore, if $[u] \in X^0(F)$, then $u_\lambda = 0$, so that $[u] \notin X^0(F')$.

To prove surjectivity, we need to show that any $[u] \in X_A$ belongs to some $X^0(F)$. We introduce the notation

$$[v_F] := [v_1 : \ldots : v_k] \text{ such that } \begin{cases} v_i = 1 & \text{if } \lambda^{(i)} \in F \\ v_i = 0 & \text{if } \lambda^{(i)} \notin F \ ; \end{cases}$$

in particular we have that $[v_P] = [1 : \ldots : 1]$. By definition of X_A, any $[u] \in X_A$ is of the form

$$[u] = \lim_{z \to 0} f(z) \cdot [v_P]$$

for some analytic map of a punctured disk given by

$$\begin{aligned} &f : \{z \in \mathbb{C}^* \mid |z| < \varepsilon\} \longrightarrow (\mathbb{C}^*)^n \\ &f(z) = (c_1 z^{a_1} + \underbrace{\ldots}_{\text{h.d.}}, \ldots, c_n z^{a_n} + \underbrace{\ldots}_{\text{h.d.}}) \ , \end{aligned}$$

for some constants $c_i \in \mathbb{C}^*$, some exponents $a_i \in \mathbb{Z}$ and where the underbraced dots represent terms of higher degree. Given $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, let $f_a : \mathbb{R}^n \to \mathbb{R}$ be the linear function defined by inner product with a, $f_a(\lambda) := a \cdot \lambda$. Let

$$F_a := \{\lambda \in P \text{ where } f_a \text{ achieves its minimum }\} \ ;$$

the set F_a is a face of P, called the **supporting face of** f_a. For simplicity, suppose that $F_a \cap A = \{\lambda^{(1)}, \lambda^{(2)}\}$; the general case is just harder for notation. Let $c = (c_1, \ldots, c_n) \in (\mathbb{C}^*)^n$. Then

$$\begin{aligned}
[u] &= \lim_{z\to 0} f(z)\cdot [v_P] \\
&= \lim_{z\to 0} [(f(z))^{\lambda^{(1)}} : \ldots : (f(z))^{\lambda^{(k)}}] \\
&= [c^{\lambda^{(1)}} + \underbrace{\ldots}_{\text{p.p.}} : c^{\lambda^{(2)}} + \underbrace{\ldots}_{\text{p.p.}} : \underbrace{\ldots}_{\text{p.p.}}] \\
&= \lim_{z\to 0} [c^{\lambda^{(1)}} : c^{\lambda^{(2)}} : 0 : \ldots : 0] \\
&= c\cdot [v_{F_a}] \in X^0(F_a)\ ,
\end{aligned}$$

where in the middle equality we have divided all homogeneous coordinates by $z^{a\cdot\lambda^{(1)}} = z^{a\cdot\lambda^{(2)}}$ and the underbraced dots represent terms with only positive powers of z. □

6.4 Fans from Polytopes

Let $P \subset \mathbb{R}^n$ be a polytope, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a linear function. We denote by $\text{supp}_P f$ the supporting face of f in P, that is, the set of points in P where f achieves its minimum, as defined in the previous section.

Definition 6.4.1. *Let F be a face of a polytope $P \subset \mathbb{R}^n$. The* **cone associated to** *F is the closure of the subset $C_{F,P} \subset (\mathbb{R}^n)^*$ consisting of all linear functions $f \in (\mathbb{R}^n)^*$ such that* $\text{supp}_P f = F$.

Exercise 57
Show that $C_{F,P}$ is a convex cone, and that the collection of cones $C_{F,P}$ for all faces of P forms a complete fan.

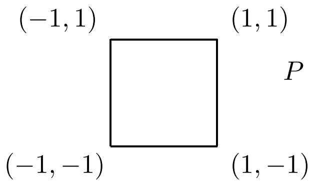
Hint: Read the description in terms of the *dual polytope*, after the next definition, and translate P so that it contains the origin in its interior.

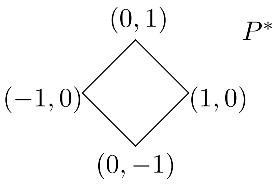
Definition 6.4.2. *The* **fan of the polytope** *P is the collection $\mathcal{F}_P$ of the cones $C_{F,P}$ for all faces F of P.*

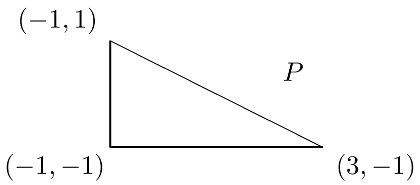
Suppose that the polytope P contains the origin in its interior. The fan of the polytope P coincides with the fan spanned by the faces of the *dual polytope*:

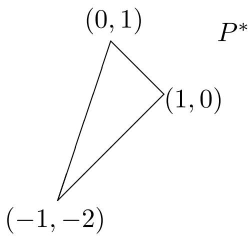
$$P^* := \{f \in (\mathbb{R}^n)^* \mid f(v) \geq -1\ ,\ \forall v \in P\}\ ,$$

that is, the collection of cones formed by the rays from the origin through the proper faces of P^*, plus the origin. For instance, the dual of a cube is an octahedron, so the fan of a cube has eight 3-dimensional triangular cones, together with all corresponding faces. Simple examples in $\mathbb{R}^2$ are:



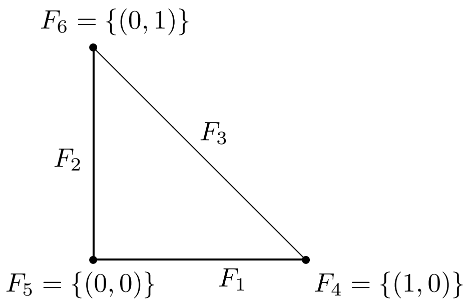




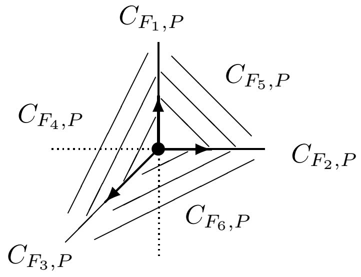


If P is rational, then (P^* is rational and) $\mathcal{F}_P$ is rational, and if P is smooth, then $\mathcal{F}_P$ is smooth.

Example. Let P be the polytope in the following picture.



The fan of P is depicted below. The cone associated with the full polytope is the origin, whereas the cones associated with each of the facets F_1, F_2 and F_3, are half-lines, and the cones associated with each of the vertices F_4, F_5 and F_6 are two-dimensional (shaded regions).



$\diamondsuit$

As in Section 4.6, consider a torus $(\mathbb{C}^*)^n$ acting linearly on a vector space V and the associated action on the projectivization $\mathbb{P}(V)$. Let v be a nonzero vector in V, and let $\overline{\mathcal{O}_v}$ be the closure in $\mathbb{P}(V)$ of the $(\mathbb{C}^*)^n$-orbit through $[v]$. Then the toric variety $\overline{\mathcal{O}_v}$ is equivalent to X_{A_v}, where $A_v = \{\lambda \in \mathbb{Z}^n \mid v_\lambda \neq 0\}$.

Proposition 6.4.3. *The fan of the toric variety $\overline{\mathcal{O}_v}$ equals the fan of the weight polytope P_v. In particular,* $\dim \overline{\mathcal{O}_v} = \dim P_v$.

For a proof of the previous proposition, see for instance [22, p.191] and [41].

Exercise 58
Check that, in a similar way, we can define a fan of a convex polyhedron, though in this case the fan may not be complete.

Remark. Let $\text{Spec}_{\text{m}}\ \mathbb{C}[S]$ be the affine toric variety associated to the finitely generated semigroup $S \subseteq \mathbb{Z}^n$. We may assume that S generates $\mathbb{Z}^n$ as an abelian group. The variety $\text{Spec}_{\text{m}}\ \mathbb{C}[S]$ is normal if and only if $S = P \cap \mathbb{Z}^n$ where P is the convex hull of S in $\mathbb{R}^n$. When $\text{Spec}_{\text{m}}\ \mathbb{C}[S]$ is normal, its fan coincides with the fan of the convex polyhedron P. $\diamondsuit$

6.5 Classes of Toric Varieties

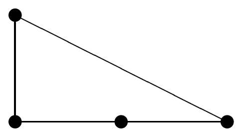
Since all smooth varieties are normal, we restrict our attention to the universe of normal toric varieties, which are classified by fans. The *affine* ones correspond to fans consisting of the set of all faces of a single n-dimensional cone (see the remark at the end of the previous section). The *compact* ones correspond to complete fans. The *equivariantly projective* ones are necessarily of the form X_A for some set of the form $A = \mathbb{Z}^k \cap P$ where P is a polytope.[2] Since projective spaces are compact (or

[2]When $A \subset \mathbb{Z}^k$ is finite yet not of the form $\mathbb{Z}^k \cap P$ for some polytope P, the corresponding X_A is not normal, and its normalization is $X_{A'}$, where $A' = \mathbb{Z}^k \cap P'$ and P' is the convex hull of A.

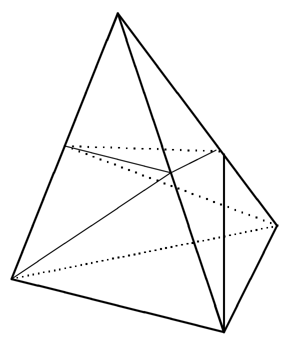
because fans of polytopes are complete), any equivariantly projective toric variety is compact.

Relation between these classes:

- Not all equivariantly projective normal toric varieties are smooth. To see a nonsmooth (i.e., singular) one just take the fan of a simple rational nonsmooth polytope. For instance, the triangle below fails the smoothness condition at the top vertex.



- Not all compact normal toric varieties are equivariantly projective, though in (complex) dimensions 1 and 2 this is always the case. Equivalently, not all complete fans come from polytopes in the sense of Definition 6.4.2, though in dimensions 1 and 2 they do. There are plenty of complete fans in $\mathbb{R}^3$ which do not come from polytopes. For example, the collection of cones over the subdivision below of the boundary of the tetrahedron is not associated to any polytope [21, 41].



- Of course, not all normal toric varieties are compact – any affine toric variety is not compact; more generally, any fan which is not complete corresponds to a noncompact toric variety.

6.6 Symplectic vs. Algebraic

A **lattice polytope** in $\mathbb{R}^n$ is a polytope whose vertices belong to $\mathbb{Z}^n$.

Suppose that Δ is an n-dimensional polytope which is both Delzant and lattice. As a Delzant polytope, it is the moment polytope of a symplectic toric manifold $(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu_\Delta)$, by Delzant's construction.

On the other hand, consider the set of integral points in Δ:

$$A := \mathbb{Z}^n \cap \Delta = \{\lambda^{(1)}, \ldots, \lambda^{(k)}\} \ ,$$

where $k := \# A$ is the number of such points. The convex hull of A is obviously Δ. Then the associated variety X_A is a toric variety for $(\mathbb{C}^*)^n$. The variety X_A is smooth and compact because the fan of the polytope Δ is smooth and complete, and X_A is connected because it is the closure of a $(\mathbb{C}^*)^n$-orbit. Moreover, by definition X_A is equivariantly embedded in $\mathbb{P}^{k-1}$,

$$i : X_A \hookrightarrow \mathbb{P}^{k-1} \ ,$$

and the restriction of the $(\mathbb{C}^*)^n$-action to its real subgroup

$$\mathbb{T}^n = \{(t_1, \ldots, t_n) \in (\mathbb{C}^*)^n \mid |t_i| = 1 \text{ for all } i \ \}$$

is effective, because the action of $(\mathbb{C}^*)^n$ was already effective.

Recall that projective spaces have canonical symplectic structures provided by the Fubini-Study forms. For later convenience, we equip $\mathbb{P}^{k-1}$ with the symplectic structure $-2\omega_{\text{FS}}$. Since X_A is a *complex* submanifold of $\mathbb{P}^{k-1}$ and ω_{FS} is a Kähler form, we obtain that the restriction

$$\omega_A := i^*(-2\omega_{\text{FS}})$$

is *nondegenerate*, hence a symplectic form on X_A. The structure ω_A is $\mathbb{T}^n$-invariant because ω_{FS} is $\mathbb{T}^n$-invariant.

We will check that the $\mathbb{T}^n$-action on X_A is in fact hamiltonian by exhibiting a moment map.

The action of $\mathbb{T}^n$ on $(\mathbb{C}^k, -2\omega_0)$ by

$$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_k) = (t^{\lambda^{(1)}} z_1, \ldots, t^{\lambda^{(k)}} z_k)$$

is hamiltonian with moment map

$$\widetilde{\mu} : \mathbb{C}^k \longrightarrow \mathbb{R}^n \ , \qquad \widetilde{\mu}(z_1, \ldots, z_k) = \sum_{j=1}^k \lambda^{(j)} |z_j|^2 \ .$$

The action of S^1 on $(\mathbb{C}^k, -2\omega_0)$ by diagonal multiplication

$$w \in S^1 \quad \longmapsto \quad \text{multiplication by} \begin{bmatrix} w & & \\ & \ddots & \\ & & w \end{bmatrix}$$

is hamiltonian with moment map

$$\phi : \mathbb{C}^k \longrightarrow \mathbb{R} \ , \qquad \phi(z) = ||z||^2 - 1 \ .$$

The manifold $(\mathbb{P}^{k-1}, -2\omega_{\text{FS}})$ is the symplectic reduction of $(\mathbb{C}^k, -2\omega_0)$ with respect to the S^1-action and the moment map ϕ. Since the S^1-action commutes with the action of $\mathbb{T}^n$ and preserves the moment map $\widetilde{\mu}$, we conclude that the $\mathbb{T}^n$-action and $\widetilde{\mu}$ descend to the quotient $(\mathbb{P}^{k-1}, -2\omega_{\text{FS}})$. Therefore, the $\mathbb{T}^n$-action on $(\mathbb{P}^{k-1}, -2\omega_{\text{FS}})$ by

$$(t_1, \ldots, t_n) \cdot [z_1 : \ldots : z_k] = [t^{\lambda^{(1)}} z_1 : \ldots : t^{\lambda^{(k)}} z_k]$$

is hamiltonian with moment map

$$\mu : \mathbb{P}^{k-1} \longrightarrow \mathbb{R}^n \ , \qquad \mu[z_1 : \ldots : z_k] = \frac{\sum_{j=1}^{k} \lambda^{(j)} |z_j|^2}{\sum_{j=1}^{k} |z_j|^2} \ .$$

The image of μ is the convex hull of A, i.e., is the polytope Δ.

As the symplectic submanifold (X_A, ω_A) is $\mathbb{T}^n$-invariant, the restriction of μ to X_A produces a moment map for the restricted action. We claim that the image of $\mu|_{X_A}$ is still Δ, so that the two constructions (Delzant's and the toric variety) yield equivalent symplectic toric manifolds.

Since $\mu(X_A)$ is a Delzant polytope, it suffices to show that each vertex of Δ is in $\mu(X_A)$. Let $\lambda^{(\ell)}$ be a vertex, and let $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ be such that the restriction to Δ of the linear function $f_a : \mathbb{R}^n \to \mathbb{R}$, $f_a(\lambda) := a \cdot \lambda$, achieves its minimum in $\lambda^{(\ell)}$. Consider the map of a punctured disk to the torus

$$\begin{array}{l} f : \{z \in \mathbb{C}^* \mid |z| < \varepsilon\} \longrightarrow (\mathbb{C}^*)^n \\ f(z) = (z^{a_1}, \ldots, z^{a_n}) \ . \end{array}$$

By definition of X_A, we have that

$$\lim_{z \to 0} [(f(z))^{\lambda^{(1)}} : \ldots : (f(z))^{\lambda^{(k)}}] \in X_A \ .$$

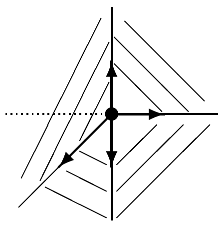
Hence, by continuity of μ, we conclude that

$$\lambda^{(\ell)} = \lim_{z \to 0} \frac{\sum \lambda^{(j)} |f(z)^{\lambda^{(j)}}|^2}{\sum |f(z)^{\lambda^{(j)}}|^2} \in \mu(X_A) \ .$$

The coincidence of the two constructions allows to see that a symplectic toric manifold is Kähler, because it inherits a compatible invariant complex structure from its equivariant embeddings into projective spaces.

Remark. Not all toric varieties admit symplectic forms. A compact normal toric variety admits a symplectic form if and only if its fan comes from some polytope. Changing the cohomology class of the symplectic form corresponds to changing the lengths of the edges of the polytope. The size of the faces of a polytope cannot be recovered from the fan, where only the combinatorics of the faces is encoded. Hence, the fan does not give the cohomology class of the symplectic form. $\diamondsuit$

Exercise 59
Find the toric variety corresponding to the fan depicted below.



Bibliography

[1] Abraham, R., Marsden, J. E., *Foundations of Mechanics*, second edition, Addison-Wesley, Reading, 1978.

[2] Arnold, V., *Mathematical Methods of Classical Mechanics*, Graduate Texts in Mathematics **60**, Springer-Verlag, New York, 1978.

[3] Atiyah, M., Convexity and commuting Hamiltonians, *Bull. London Math. Soc.* **14** (1982), 1-15.

[4] Atiyah, M., Bott, R., The moment map and equivariant cohomology, *Topology* **23** (1984), 1-28.

[5] Atiyah, M., Macdonald, I., *Introduction to Commutative Algebra*, Addison-Wesley, Reading, 1969.

[6] Audin, M., *The Topology of Torus Actions on Symplectic Manifolds*, Progress in Mathematics **93**, Birkhäuser Verlag, Basel, 1991.

[7] Audin, M., *Spinning Tops. A Course on Integrable Systems*, Cambridge Studies in Advanced Mathematics **51**, Cambridge University Press, Cambridge, 1996.

[8] Batyrev, V., Mirror symmetry and toric geometry, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), *Doc. Math.* **1998**, Extra Vol. II, 239-248.

[9] Bott, R., Tu, L., *Differential Forms in Algebraic Topology*, Graduate Texts in Mathematics **82**, Springer-Verlag, New York-Berlin, 1982.

[10] Bredon, G., *Introduction to Compact Transformation Groups*, Pure and Applied Mathematics **46**, Academic Press, New York-London, 1972.

[11] Bröcker, T., tom Dieck, T., *Representations of Compact Lie Groups*, Graduate Texts in Mathematics **98**, Springer-Verlag, New York, 1985.

[12] Cannas da Silva, A., *Lectures on Symplectic Geometry*, Lecture Notes in Mathematics **1764**, Springer-Verlag, Berlin, 2001.

[13] Cox, D., Recent developments in toric geometry, *Algebraic Geometry – Santa Cruz 1995*, 389-436, Proc. Sympos. Pure Math. **62**, part 2, Amer. Math. Soc., Providence, 1997.

[14] Danilov, V., The geometry of toric varieties, *Uspekhi Mat. Nauk* **33** (1978), no. 2 (200), 85-134, 247, English translation: *Russian Math. Surveys* **33** (1978), no. 2, 97-154.

[15] Delzant, T., Hamiltoniens périodiques et images convexes de l'application moment, *Bull. Soc. Math. France* **116** (1988), 315-339.

[16] Demazure, M., Sous-groupes algébriques de rang maximum du groupe de Cremona, *Ann. Sci. École Norm. Sup.* (4) **3** (1970), 507-588.

[17] Duistermaat, J.J., On global action-angle coordinates, *Comm. Pure Appl. Math.* **33** (1980), 687-706.

[18] Duistermaat, J.J., Equivariant cohomology and stationary phase, *Symplectic Geometry and Quantization* (Sanda and Yokohama, 1993), edited by Maeda, Y., Omori, H. and Weinstein, A., 45-62, *Contemp. Math.* **179**, Amer. Math. Soc., Providence, 1994.

[19] Duistermaat, J.J., Heckman, G., On the variation in the cohomology of the symplectic form of the reduced phase space, *Invent. Math.* **69** (1982), 259-268; Addendum, *Invent. Math.* **72** (1983), 153-158.

[20] Ewald, G., *Combinatorial Convexity and Algebraic Geometry*, Graduate Texts in Mathematics **168**, Springer-Verlag, New York, 1996.

[21] Fulton, W., *Introduction to Toric Varieties*, Annals of Mathematics Studies **131**, Princeton University Press, Princeton, 1993.

[22] Gelfand, I., Kapranov, M., Zelevinsky, A., *Discriminants, Resultants, and Multidimensional Determinants*, Mathematics: Theory & Applications, Birkhäuser Boston, 1994.

[23] Griffiths, P., Harris, J., *Principles of Algebraic Geometry*, Chapter 0, reprint of the 1978 original, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994.

[24] Guillemin, V., *Moment Maps and Combinatorial Invariants of Hamiltonian* T^n*-spaces*, Progress in Mathematics **122**, Birkhäuser, Boston, 1994.

[25] Guillemin, V., Sternberg, S., Convexity properties of the moment mapping, *Invent. Math.* **67** (1982), 491-513.

[26] Guillemin, V., Sternberg, S., Birational equivalence in the symplectic category, *Invent. Math.* **97** (1989), 485-522.

[27] Guillemin, V., Sternberg, S., *Symplectic Techniques in Physics*, second edition, Cambridge University Press, Cambridge, 1990.

[28] Kempf, G., Knudsen, F., Mumford, D., Saint-Donat, B., *Toroidal Embeddings, I*, Lecture Notes in Mathematics **339**, Springer-Verlag, Berlin-New York, 1973.

[29] Kirwan, F., *Cohomology of Quotients in Symplectic and Algebraic Geometry*, Mathematical Notes **31**, Princeton University Press, Princeton, 1984.

[30] Kodaira, K., On the structure of compact complex analytic surfaces, I, *Amer. J. Math.* **86** (1964), 751-798.

[31] Lerman, E., Symplectic cuts, *Math. Res. Lett.* **2** (1995), 247-258.

[32] Lerman, E., Meinrenken, E., Tolman, S., Woodward, C., Nonabelian convexity by symplectic cuts, *Topology* **37** (1998), 245-259.

[33] Lerman, E., Tolman, S., Hamiltonian torus actions on symplectic orbifolds and toric varieties, *Trans. Amer. Math. Soc.* **349** (1997), 4201-4230.

[34] Marsden, J., Weinstein, A., Reduction of symplectic manifolds with symmetry, *Rep. Mathematical Phys.* **5** (1974), 121-130.

[35] McDuff, D., Examples of simply-connected symplectic non-Kählerian manifolds, *J. Differential Geom.* **20** (1984), 267-277.

[36] McDuff, D., Salamon, D., *Introduction to Symplectic Topology*, Oxford Mathematical Monographs, Oxford University Press, New York, 1995.

[37] Meyer, K., Symmetries and integrals in mechanics, *Dynamical Systems* (Proc. Sympos., Univ. Bahia, Salvador, 1971), 259-272. Academic Press, New York, 1973.

[38] Mikhalkin, G., Amoebas of algebraic varieties, survey for the Real Algebraic and Analytic Geometry Congress, Rennes, June 11-15, 2001, preprint available at `http://xxx.lanl.gov/abs/math.AG/0108225`.

[39] Milnor, J., *Morse Theory*, based on lecture notes by M. Spivak and R. Wells, Annals of Mathematics Studies **51**, Princeton University Press, Princeton, 1963.

[40] Morse, M., The foundations of a theory in the calculus of variations in the large, *Trans. Amer. Math. Soc.* **30** (1928), 213-274.

[41] Oda, T., *Convex Bodies and Algebraic Geometry – An Introduction to the Theory of Toric Varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **15**, Springer-Verlag, Berlin, 1988.

[42] Satake, I., On a generalization of the notion of manifold, *Proc. Nat. Acad. Sci. U.S.A.* **42** (1956), 359-363.

[43] Scott, P., The geometries of 3-manifolds, *Bull. London Math. Soc.* **15** (1983), 401-487.

[44] Shafarevich, I., *Basic Algebraic Geometry, 1, Varieties in Projective Space*, 2nd edition, translated from the 1988 russian edition and with notes by Miles Reid, Springer-Verlag, Berlin, 1994.

[45] Souriau, J.-M. , *Structure des Systèmes Dynamiques*, Maîtrises de Mathématiques, Dunod, Paris 1970.

[46] Spivak, M., *A Comprehensive Introduction to Differential Geometry*, Vol. I, second edition, Publish or Perish, Inc., Wilmington, 1979.

[47] Sturmfels, B., *Gröbner Bases and Convex Polytopes*, University Lecture Series **8**, Amer. Math. Soc., Providence, 1996.

[48] Weinstein, A., *Lectures on Symplectic Manifolds*, Regional Conference Series in Mathematics **29**, Amer. Math. Soc., Providence, 1977.

[49] Wells, R.O., *Differential Analysis on Complex Manifolds*, second edition, Graduate Texts in Mathematics **65**, Springer-Verlag, New York-Berlin, 1980.

[50] Zariski, O., Samuel, P., *Commutative Algebra*, Vol. 1, with the cooperation of I.S. Cohen, Graduate Texts in Mathematics **28**, Springer-Verlag, New York-Heidelberg-Berlin, 1975.