GL VERLINDE NUMBERS AND THE GRASSMANN TQFT

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1. INTRODUCTION

These notes are concerned with moduli spaces of bundles on a smooth projective curve. Over them we consider determinant line bundles and their holomorphic Euler characteristics, the Verlinde numbers. The goal is to give a brief exposition of the two-dimensional topological quantum field theory that captures the structure of the GL Verlinde numbers, associated with spaces of bundles with varying determinant. Our point of view is to emphasize the close connection with another TQFT, the quantum cohomology of the Grassmannian.

Two different geometries are related here, the moduli of bundles on a curve C and the space of maps from C to a suitable Grassmannian. The connection between them was established in the classic paper [W] where the open and closed invariants of the GL Verlinde TQFT, in all genera, were exhaustively written in both geometries. On the mathematical side, it was shown [A] that the underlying algebras of the two TQFTs are isomorphic, as the genus zero three-point invariants match. The TQFTs turn up different invariants overall, due to a discrepancy in the metrics of the associated Frobenius algebras. Moreover, the higher genus GL Verlinde invariants, open or closed, have not been systematically written down in the mathematics literature although they were shown in [W] to have compelling closed-form geometric expressions. We found it useful therefore to render the results of [W] in standard mathematical language, also with a view toward future studies of q-deformations of ordinary two-dimensional Yang Mills theory.

The exposition is organized as follows. After briefly recalling the notion of a two-dimensional TQFT in the next section, we introduce in our context, on a smooth projective curve C, the two spaces of interest: the ancestor of all moduli spaces of sheaves, the Grothendieck Quot scheme, and the moduli space of semistable bundles. We present the former here primarily as compactifying the space of maps from the curve to a Grassmannian. Relevant aspects of the geometry and intersection theory of the two spaces are discussed. The last section studies the relation between them, in the form of the GL Verlinde TQFT, which we also refer to as the Grassmann TQFT.

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2. Generalities on two-dimensional TQFTs

We consider the category **2Cob**, in which

(i) the objects are one-dimensional compact oriented manifolds *i.e.*, finite unions of oriented circles;
(ii) the morphisms are (diffeomorphism classes of) oriented cobordisms;
(iii) composition of morphisms is concatenation of cobordisms;
(iv) there is a tensor structure given by taking disjoint unions of objects.

Let $\mathbf{Vect}_{\mathbb{C}}$ be the category of $\mathbb{C}$-vector spaces. A two-dimensional $\mathbb{C}$-valued TQFT is a symmetric monoidal functor

$$F: \ \mathbf{2Cob} \longrightarrow \mathbf{Vect}_{\mathbb{C}}.$$

There is a basic vector space H in the theory, representing the value of the functor F at the oriented circle S^1. In addition, F associates to the empty manifold the vector space $\mathbb{C}$.

The datum of the functor is equivalent to the structure of a commutative Frobenius algebra on H. By definition this comprises

(i) a commutative associative multiplication
$$H \otimes H \xrightarrow{\bullet} H$$
with identity element, and
(ii) a symmetric nondegenerate pairing
$$(\cdot\,,\,\cdot) : H \otimes H \to \mathbb{C}$$
satisfying the Frobenius property
$$(a \cdot b,\, c) = (a,\, b \cdot c).$$

Indeed, if $W_s^t(g)$ is the genus g cobordism with s inputs and t outputs, then

(i) $F(W_2^1(0)) : H \otimes H \to H$ is the algebra multiplication,
(ii) $F(W_0^1(0)) : \mathbb{C} \to H$ is the identity element,
(iii) $F(W_2^0(0))$ gives the pairing $(\cdot\,,\,\cdot)$.

Viewed as a cobordism from the empty manifold to the empty manifold, a closed surface of genus g corresponds under F to a homomorphism from $\mathbb{C}$ to $\mathbb{C}$, thus to a number $F(g)$,
$$F(g) = F(W_0^0(g)).$$

Let us assume that H has a preferred basis,
$$H = \bigoplus_{\lambda} \mathbb{C} e_\lambda.$$

The vector space $H^{\otimes s}$ has a basis $e_{\underline{\lambda}}$ indexed by multi-indices $\underline{\lambda} = (\lambda_1, \dots, \lambda_s)$:

$$e_{\underline{\lambda}} = e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_s}.$$

We denote by $F(g)^{\underline{\mu}}_{\underline{\lambda}}$ the matrix entries of the cobordism homomorphism

$$F(W^t_s(g)) : H^{\otimes s} \longrightarrow H^{\otimes t}$$

in this basis. We thus have

$$F(W^t_s(g)) : \ e_{\underline{\lambda}} \mapsto F(g)^{\underline{\mu}}_{\underline{\lambda}}\, e_{\underline{\mu}},$$

where $\underline{\lambda}$, $\underline{\mu}$ are multi-indices (with s and t components respectively). The TQFT is equivalent to the data of the numbers $F(g)^{\underline{\mu}}_{\underline{\lambda}}$ satisfying gluing rules which reflect the functoriality,

$$\sum_{\underline{\mu}} F(g_1)^{\underline{\mu}}_{\underline{\lambda}}\, F(g_2)^{\underline{\nu}}_{\underline{\mu}} = F(g_1 + g_2 + t - 1)^{\underline{\nu}}_{\underline{\lambda}}. \tag{1}$$

Here t is the number of components of the multi-index $\underline{\mu}$, which is summed over.

3. The Quot scheme $Q_C(\mathbb{G}(r,n),d)$

Let C be a smooth complex projective curve of genus g. We let $Q_C(\mathbb{G}(r,n),d)$ denote the Grothendieck Quot scheme parametrizing rank $n-r$ degree d quotients of the rank n trivial sheaf on C. A point in the Quot scheme is given by a short exact sequence

$$0 \to E \to \mathcal{O}_C \otimes \mathbb{C}^n \to F \to 0.$$

While the kernel sheaf E is always locally free, the quotient F is in general a sum

$$F = \overline{F} \oplus T,$$

with $\overline{F}$ locally free and T a torsion sheaf supported at finitely many points of the curve C.

The quotients F which *are* locally free form an open locus in $Q_C(\mathbb{G}(r,n),d)$, and can be regarded as degree d maps

$$f : C \to \mathbb{G}(r,n)$$

from C to the Grassmannian $\mathbb{G}(r,n)$ of r planes in $\mathbb{C}^n$. The Quot scheme may be viewed as compactifying the space $\mathrm{Mor}_d(C, \mathbb{G}(r,n))$ of degree d maps to $\mathbb{G}(r,n)$:

$$\mathrm{Mor}_d(C, \mathbb{G}(r,n)) \hookrightarrow Q_C(\mathbb{G}(r,n),d).$$

3.1. **Examples.** When $C = \mathbb{P}^1$ and $r = 1$, the Quot scheme $Q_{\mathbb{P}^1}(\mathbb{P}^{n-1}, d)$ is the projectivized space of n homogeneous degree d polynomials in $\mathbb{C}[x, y]$,

$$Q_{\mathbb{P}^1}(\mathbb{P}^{n-1}, d) \simeq \mathbb{P}^{n(d+1)-1}.$$

In general, when $r = 1$ and C has arbitrary genus, $Q_C(\mathbb{P}^{n-1}, d)$ parametrizes exact sequences

$$0 \to L \to \mathcal{O}_C \otimes \mathbb{C}^n \to Q \to 0$$

where L is a line bundle of degree $-d$. Equivalently, dualizing such exact sequences, points in the space are degree d line bundles $L^\vee$ on C together with n sections, not all zero:

$$\mathcal{O}_C \otimes \mathbb{C}^{n\vee} \to L^\vee.$$

Let $\mathrm{Jac}^d(C)$ be the Picard variety of degree d line bundles on C, and let

$$\pi : \mathrm{Jac}^d(C) \times C \to \mathrm{Jac}^d(C)$$

be the projection. For d sufficiently large, $d \geq 2g - 1$, the push forward $\pi_\star \mathcal{P}$ of the Poincaré line bundle

$$\mathcal{P} \to \mathrm{Jac}^d(C) \times C$$

is locally free, and its fiber over $[L] \in \mathrm{Jac}^d(C)$ is the space $H^0(C, L)$ of sections of L. In this case,

$$Q_C(\mathbb{P}^{n-1}, d) \simeq \mathbb{P}((\pi_\star \mathcal{P})^{\oplus n}) \to \mathrm{Jac}^d(C).$$

Although for arbitrary r the Quot scheme does not have such a simple description, it remains true that the space is well-behaved in the regime of large degrees d:

Theorem 1. [BDW] *For $d >> r, n, g$, the space $Q_C(\mathbb{G}(r, n), d)$ is irreducible, generically smooth, and has the expected dimension.*

3.2. **Structures.** As a fine moduli space, the Quot scheme carries a universal sequence

$$0 \to \mathcal{S} \to \mathcal{O} \otimes \mathbb{C}^n \to \mathcal{Q} \to 0 \;\text{ on }\; Q_C(\mathbb{G}(r, n), d) \times C,$$

with the universal subsheaf $\mathcal{S}$ being locally free. The tangent sheaf to $Q_C(\mathbb{G}(r, n), d)$ is given as

$$\mathcal{T} Q_C(\mathbb{G}(r, n), d) \simeq \mathrm{Hom}_\pi(\mathcal{S}, \mathcal{Q}),$$

where

$$\pi : Q_C(\mathbb{G}(r, n), d) \times C \to Q_C(\mathbb{G}(r, n), d)$$

is the projection. The obstruction sheaf is $\mathrm{Ext}^1_\pi(\mathcal{S}, \mathcal{Q})$. The expected dimension is

$$e = nd - r(n - r)(g - 1)$$

by the Riemann-Roch formula.

The Chern classes of the universal subsheaf are natural to consider for the intersection theory of $Q_C(\mathbb{G}(r,n),d)$. Fixing a basis

$$1, \delta_1, \ldots, \delta_{2g}, \omega$$

for the cohomology of the curve C, we write

$$c_k(\mathcal{S}^\vee) = a_k \otimes 1 + \sum_{i=1}^{2g} b_k^i \otimes \delta_i + f_k \otimes \omega, \quad 1 \le k \le r,$$

where

$$a_k \in H^{2k}(Q_C(\mathbb{G}(r,n),d),\mathbb{C}), \;\; b_k^i \in H^{2k-1}(Q_C(\mathbb{G}(r,n),d),\mathbb{C}),$$
$$f_k \in H^{2k-2}(Q_C(\mathbb{G}(r,n),d),\mathbb{C}).$$

Note that

$$f_k = \pi_\star c_k(\mathcal{S}^\vee), \tag{2}$$

while for $p \in C$ and

$$\mathcal{S}_p = \mathcal{S}|_{Q_C(\mathbb{G}(r,n),d)\times\{p\}},$$

we have

$$a_k = c_k(\mathcal{S}_p^\vee). \tag{3}$$

When d is large so that $Q_C(\mathbb{G}(r,n),d)$ is irreducible, top intersections of the tautological a, b and f classes can be evaluated meaningfully against the fundamental class. For arbitrary degrees, the Quot scheme may be reducible and oversized. However, intersection theory can still be pursued in a virtual sense, by pairing Chern classes against a virtual fundamental cycle of the expected dimension, which the Quot scheme possesses:

Theorem 2. [CFK], [MO1] *The Quot scheme $Q_C(\mathbb{G}(r,n),d)$ has a two-term perfect obstruction theory and a virtual fundamental class of expected dimension*

$$[Q_C(\mathbb{G}(r,n),d)]^{vir} \in A_e(Q_C(\mathbb{G}(r,n),d)).$$

Proof. We show that the tangent-obstruction complex for $Q_C(\mathbb{G}(r,n),d)$ admits a resolution

$$0 \to \mathrm{Hom}_\pi(\mathcal{S},\mathcal{Q}) \to \mathcal{A}_0 \to \mathcal{A}_1 \to \mathrm{Ext}^1_\pi(\mathcal{S},\mathcal{Q}) \to 0, \tag{4}$$

where the sheaves $\mathcal{A}_0$ and $\mathcal{A}_1$ are locally free. The virtual fundamental class is then standardly constructed as described in [LT], using the two vector bundles $\mathcal{A}_0, \mathcal{A}_1$.

The resolution is easily obtained as follows. Let $\mathcal{O}(1)$ be a degree 1 line bundle on the curve C, and denote by $\mathcal{S}(m)$, $\mathcal{Q}(m)$ the twists of the tautological sheaves by the pullback of $\mathcal{O}(m)$ on C to the product $Q_C(\mathbb{G}(r,n),d) \times C$. Let m be large enough so that

$$R^1\pi_\star\mathcal{S}(m) = R^1\pi_\star\mathcal{Q}(m) = 0,$$

and so that the evaluation map

$$\pi^\star\left(R^0\pi_\star\mathcal{S}(m)\right)\to\mathcal{S}(m)$$

is surjective. The pushforward sheaves $R^0\pi_\star\mathcal{S}(m), R^0\pi_\star\mathcal{Q}(m)$ are then locally free. Further let $\mathcal{K}$ be the kernel

$$0\to\mathcal{K}\to\pi^\star\left(R^0\pi_\star\mathcal{S}(m)\right)\otimes\mathcal{O}(-m)\to\mathcal{S}\to 0.$$

Applying the functor $\mathrm{Hom}_\pi(\cdot,\mathcal{Q})$ gives

$$0\to\mathrm{Hom}_\pi(\mathcal{S},\mathcal{Q})\to\left(R^0\pi_\star\mathcal{S}(m)\right)^\vee\otimes R^0\pi_\star\mathcal{Q}(m)\to\mathrm{Hom}_\pi(\mathcal{K},\mathcal{Q})\to$$

$$\to\mathrm{Ext}^1_\pi(\mathcal{S},\mathcal{Q})\to 0.$$

Continuing this sequence one more term we get $\mathrm{Ext}^1_\pi(\mathcal{K},\mathcal{Q})=0$, so the sheaf

$$\mathcal{A}_1=_{\mathrm{def}}\mathrm{Hom}_\pi(\mathcal{K},\mathcal{Q})$$

is locally free. Also,

$$\mathcal{A}_0=_{\mathrm{def}}\left(R^0\pi_\star\mathcal{S}(m)\right)^\vee\otimes R^0\pi_\star\mathcal{Q}(m),$$

is locally free. □

3.3. **Intersections.** In this section, we will consider the (virtual) intersection theory of Quot schemes.

We start by pointing out the compatibility of the virtual fundamental class with the natural embedding, for $p\in C$,

$$\iota_p:\ Q_C(\mathbb{G}(r,n),d)\ \hookrightarrow\ Q_C(\mathbb{G}(r,n),d+r),$$

given by

$$\{E\hookrightarrow\mathcal{O}_C\otimes\mathbb{C}^n\}\ \mapsto\ \{E(-p)\to E\to\mathcal{O}_C\otimes\mathbb{C}^n\}.$$

A degree $-d-r$ subsheaf

$$E'\hookrightarrow\mathcal{O}_C\otimes\mathbb{C}^n$$

comes from $Q_C(\mathbb{G}(r,n),d)$ if the dual map

$$\mathcal{O}_C\otimes\mathbb{C}^{n\vee}\to E'^\vee$$

is zero at p. The image of the degree d Quot scheme inside the degree $d+r$ space is therefore the zero locus of the dual universal map

$$\mathcal{O}\otimes\mathbb{C}^{n\vee}\to\mathcal{S}_p^\vee\ \text{ on }\ Q_C(\mathbb{G}(r,n),d+r).$$

This relationship is reflected on the level of the virtual fundamental classes for the two spaces. We recall that a_r is the top Chern class of the universal subsheaf $\mathcal{S}_p^\vee$ before noting that

Proposition 1. [MO1] *The equality*

$$\iota_{p_\star}[Q_C(\mathbb{G}(r,n),d)]^{vir} = a_r^n \cap [Q_C(\mathbb{G}(r,n),d+r)]^{vir} \tag{5}$$

holds in $A_\star(Q_C(\mathbb{G}(r,n),d+r))$.

The intersection theory of a-classes is well understood. Top intersections are given in closed form by the *Vafa-Intriligator formula.* Furthermore, in the large-degree regime, the intersection numbers express counts of maps from the curve C to the Grassmannian $\mathbb{G}(r,n)$, satisfying incidence constraints. More precisely, we have:

Theorem 3. *(i)* [Ber], [ST], [MO1] *Let* $J(x_1,\ldots,x_r)$ *be the symmetric function*

$$J(x_1,\ldots,x_r) = n^r \cdot x_1^{-1} \cdots x_r^{-1} \prod_{1\le i<j\le r} (x_i - x_j)^{-2}.$$

Let $P(a_1,\ldots,a_r)$ *be a top degree polynomial in the Chern classes of* $\mathcal{S}_p^\vee$*. Then*

$$\int_{[Q_C(\mathbb{G}(r,n),d)]^{vir}} P(a_1,\ldots,a_r) = u \cdot \sum_{\lambda_1,\ldots,\lambda_r} R(\lambda_1,\ldots,\lambda_r)\, J^{g-1}(\lambda_1,\ldots,\lambda_r),$$

where R *is the symmetric polynomial obtained by expressing* $P(a_1,\ldots a_r)$ *in terms of the Chern roots of* $\mathcal{S}_p^\vee$*. The sum is taken over all* $\binom{n}{r}$ *tuples*

$$(\lambda_1,\ldots,\lambda_r)$$

of distinct n*-roots of* 1*. Here*

$$u = (-1)^{(g-1)\binom{r}{2}+d(r-1)}.$$

(ii) [Ber] *When* $Q_C(\mathbb{G}(r,n),d)$ *is irreducible of the expected dimension, the above intersection counts the number of degree* d *maps from the curve* C *to* $\mathbb{G}(r,n)$ *sending fixed distinct points of* C *to special Schubert subvarieties of the Grassmannian, each Schubert variety matching an appearance of an* a*-class in the top monomial* P.

The intersection numbers appearing in Theorem 3 were written down in [I]. Mathematical proofs have relied either on degenerations of the Quot scheme to genus zero, or on equivariant localization. Degeneration arguments use the enumerativeness of the a-intersections in the large-degree situation.

By contrast, intersections involving f-classes do not give actual counts of maps, and explicit formulas for them have been relatively little explored. To describe one such formula, we let

$$\sigma_i(x) = \sigma_i(x_1,\ldots,x_r) \text{ and } \sigma_{i;k}(x) = \sigma_{i;k}(x_1,\ldots,x_r)$$

be the i^{th} elementary symmetric functions in the variables

$$x_1,\ldots,x_r \text{ and } x_1,\ldots,\widehat{x}_k,\ldots,x_r$$

respectively. In the second set of variables, x_k is omitted.

Theorem 4. [MO1] *Letting $\mathcal{D}_l, 2 \leq l \leq r$, be the first-order differential operator*

$$\mathcal{D}_l = (g-1)(r-l+1)(n-r+l-1) \cdot \sigma_{l-1}(x) + \sum_{k=1}^{r} \sigma_{l-1;k}(x)\, x_k \cdot \frac{\partial}{\partial x_k},$$

we have

$$\int_{[\mathrm{Q}_C(\mathbb{G}(r,n),d)]^{vir}} f_l \cdot P(a_1, \ldots, a_r) = \frac{u}{n} \sum_{\lambda_1,\ldots,\lambda_r} (\mathcal{D}_l R)(\lambda_1, \ldots, \lambda_r) \cdot J^{g-1}(\lambda_1, \ldots, \lambda_r).$$

The sum is over all $\binom{n}{r}$ tuples $(\lambda_1, \ldots, \lambda_r)$ of distinct n-roots of 1*.*

It would be very interesting to generalize the Vafa-Intriligator formula to include all intersections of f and a-classes.

We turn now to a discussion of the second geometry of interest.

4. The moduli space of semistable bundles

4.1. **Basics.** We consider vector bundles of rank r and degree d on the smooth curve C. We recall briefly the main facts in the moduli theory of semistable vector bundles on C. The family of *all* vector bundles of fixed topological type is not bounded, as one can immediately verify looking at vector bundles on $\mathbb{P}^1$. A notion of stability is required to get a bounded problem. For any vector bundle E, its *slope* $\mu(E)$ is defined as the ratio

$$\mu(E) = \frac{\text{degree}(E)}{\text{rank}(E)}.$$

A vector bundle E is said *stable* (*semistable*) if for all subbundles $F \hookrightarrow E$,

$$\mu(F) < \mu(E) \;\; (\mu(F) \leq \mu(E)).$$

It follows easily that

Lemma 1. *(i) If E is semistable with $\mu(E) \geq 2g-1$, then $H^1(E) = 0$.*
(ii) If E is semistable with $\mu(E) \geq 2g$, then the evaluation map of sections

$$H^0(E) \otimes \mathcal{O}_C \to E$$

is surjective.

Proof: Indeed, by Serre duality, $H^1(E) \simeq H^0(E^\vee \otimes K_C)^\vee$, where K_C denotes the canonical bundle. Let $L \hookrightarrow K_C$ be the image of an assumed nonzero homomorphism $\phi : E \to K_C$. E is semistable and L is a *quotient* of E, so we must have

$$\mu(E) \leq \mu(L) = \deg(L) \leq \deg(K_C) = 2g-2.$$

This contradicts the assumption, so there are no nonzero such homomorphisms and $H^1(E) = 0$. Regarding (ii), for any $p \in C$, taking cohomology for the sequence

$$0 \to E(-p) \to E \to E_p \to 0,$$

and using the vanishing of (i), it follows that the fiber of E at p is generated by global sections. $\square$

Fixing a line bundle $\mathcal{O}(1)$ of degree 1 on C, there is therefore an integer m such that for all semistable rank r and degree d vector bundles E, we have

$$H^1(E(m)) = 0 \text{ and } H^0(E(m)) \otimes \mathcal{O}_C \to E(m) \to 0.$$

Any semistable E can be thus realized as a quotient

$$\mathcal{O}_C^{\oplus q}(-m) \to E \to 0, \text{ with } q = \chi(E(m)),$$

i.e., as a point in the Quot scheme

$$\mathrm{Quot}_C^{r,d}(\mathcal{O}_C^{\oplus q}(-m))$$

of quotients of $\mathcal{O}_C^{\oplus q}(-m)$ of rank r and degree d. The group $SL(q)$ acts on this Quot scheme, with a standard linearization. On the locus of vector bundle quotients E in $\mathrm{Quot}_C^{r,d}(\mathcal{O}_C^{\oplus q}(-m))$ for which the quotient map induces an isomorphism

$$H^0(\mathcal{O}_C^{\oplus q}) \simeq H^0(E(m)),$$

stability in the geometric invariant theory sense coincides with slope stability. Restricting further to semistable quotients, we have an $SL(q)$-invariant subscheme

$$\mathrm{Quot}^{ss} \subset \mathrm{Quot}_C^{r,d}(\mathcal{O}_C^{\oplus q}(-m)).$$

The GIT quotient

$$\mathrm{Quot}^{ss}//SL(q) =_{\mathrm{def}} U_C(r,d)$$

is an irreducible normal projective variety of dimension $r^2(g-1)+1$, the moduli space of semistable vector bundles of rank r and degree d. The open subset

$$U_C^s(r,d) \subset U_C(r,d)$$

parametrizing isomorphism classes of *stable* vector bundles is smooth and its complement has codimension at least 2 in $U_C(r,d)$. For details on this standard construction, we refer the reader to [LeP].

4.2. **Line bundles on the moduli space and their Euler characteristics.** Twisting vector bundles by a line bundle of degree 1 on C gives an isomorphism

$$U_C(r,d) \cong U_C(r,d+r),$$

so the dependence on degree is only modulo r. We assume further for simplicity that

$$d = 0.$$

All constructions can be easily duplicated in the arbitrary degree situation.

When $r = 1$, we have
$$U_C(1,0) \simeq \mathrm{Jac}(C),$$
the Picard variety of degree 0 line bundles on C. Note that for a fixed line bundle M on C of degree $g-1$,
$$\chi(L \otimes M) = 0 \text{ for } L \in \mathrm{Jac}(C).$$
The classical theta divisor relative to M is defined as
$$\Theta_{1,M} = \{L \in \mathrm{Jac}(C) \text{ such that } h^0(L \otimes M) \neq 0\}.$$
Sections of the tensor powers of the line bundle $\mathcal{O}(\Theta_{1,M})$ are the classical theta functions, and
$$h^0(\mathrm{Jac}(C), \mathcal{O}(k\,\Theta_{1,M})) = \chi(\mathrm{Jac}(C), \mathcal{O}(k\,\Theta_{1,M})) = k^g \tag{6}$$
is the dimension of the space of level k theta functions.

For $r > 1$, we have similarly, when M is as before a line bundle of degree $g-1$ on C,
$$\chi(E \otimes M) = 0 \text{ for } E \in U_C(r,0),$$
and we set
$$\Theta_{r,M} = \{E \in U_C(r,0) \text{ such that } h^0(E \otimes M) \neq 0\}. \tag{7}$$

As in the $r = 1$ case in fact, the divisor $\Theta_{r,M}$ has a determinantal scheme structure: for a family
$$\mathcal{E} \to S \times C$$
of semistable rank r degree 0 vector bundles, flat over S, we consider a resolution
$$0 \to R^0\pi_\star(\mathcal{E} \otimes p_C^\star M) \to \mathcal{F}_0 \xrightarrow{\varphi} \mathcal{F}_1 \to R^1\pi_\star(\mathcal{E} \otimes p_C^\star M) \to 0$$
of the direct image complex
$$R\pi_\star(\mathcal{E} \otimes p_C^\star M),$$
so that $\mathcal{F}_0, \mathcal{F}_1$ are locally free. Here we denoted by π and p_C the projections
$$S \times C \xrightarrow{\pi} S, \;\; S \times C \xrightarrow{p_C} C.$$
The pullback of $\Theta_{r,M}$ to S is then the degeneracy locus of φ. The line bundle $\mathcal{O}(\Theta_{r,M})$ is the descent of the determinant line bundle
$$\det R\pi_\star(\mathcal{E} \otimes p_C^\star M)^{-1}$$
from the Quot scheme $\mathrm{Quot}_C^{r,d}(\mathcal{O}_C^{\oplus q}(-m))$, with $\mathcal{E}$ being the universal quotient.

The Picard group of $U_C(r,0)$, described in [DN], is generated by the theta line bundles $\mathcal{O}(\Theta_{r,M})$ as M varies in $\mathrm{Pic}^{g-1}(C)$, together with classical theta line bundles $\mathcal{O}(\Theta_{1,M})$ on $\mathrm{Jac}(C)$. The latter are pulled back to $U_C(r,0)$ via the morphism

$$\det : U_C(r,0) \to \mathrm{Jac}(C)$$

sending bundles to their determinants. More precisely,

Theorem 5. [DN] *(i) Consider*

$$\iota : SU_C(r,\mathcal{O}) \hookrightarrow U_C(r,0)$$

the moduli space of bundles with trivial determinant. The restriction

$$\mathcal{L} =_{def} \iota^\star \mathcal{O}(\Theta_{r,M}),$$

is independent of the choice of M in $Pic^{g-1}(C)$ and

$$Pic(SU_C(r,\mathcal{O})) \cong \mathbb{Z}\,\mathcal{L}.$$

(ii)

$$Pic(U_C(r,0)) \cong \mathbb{Z}\,\mathcal{O}(\Theta_{r,M}) \oplus \det{}^\star(Pic\,(Jac(C))).$$

As in the classical case, the theta bundles on $U_C(r,0)$ and $SU_C(r,\mathcal{O})$ have no higher cohomology, so their holomorphic Euler characteristics give also the dimension of their spaces of sections. Explicit expressions for them, known as *Verlinde formulas*, were derived by several methods, and are significantly more complicated than (6). The formulas are very similar for k powers of $\mathcal{L}$ on $SU_C(r,\mathcal{O})$ and of $\mathcal{O}(\Theta_{r,M})$ on $U_C(r,0)$. A slightly simpler and more convenient expression arises however for the twist

$$\mathcal{O}(k\,\Theta_r) \otimes \det{}^\star \mathcal{O}(\Theta_1) \in \mathrm{Pic}(U_C(r,0)).$$

Here we suppressed reference degree $g-1$ line bundles for the theta bundles, as the holomorphic Euler characteristic is independent of these choices. Writing also, to simplify notation, Θ_r and Θ_1 for the line bundles $\mathcal{O}(\Theta_r)$ and $\mathcal{O}(\Theta_1)$, we have

$$\begin{aligned} V_g^{r,k} &=_{\mathrm{def}} h^0(U_C(r,0), \Theta_r^k \otimes \det{}^\star\Theta_1) = \chi(U_C(r,0), \Theta_r^k \otimes \det{}^\star\Theta_1) \\ &= \sum_{\substack{S\sqcup T=\{1,\ldots,r+k\} \\ |S|=r}} \prod_{\substack{s\in S \\ t\in T}} \left| 2\sin \pi \frac{s-t}{r+k} \right|^{g-1}. \end{aligned} \tag{8}$$

The sum is over the $\binom{r+k}{r}$ partitions of the first $r+k$ natural numbers into two disjoint subsets S and T of cardinalities r and k. Note that the numbers $V_g^{r,k}$ depend solely on the genus g of C, the rank r, and the level k.

4.3. **Parabolic counterparts.** We would like to formulate degeneration rules for the Verlinde numbers $V_g^{r,k}$. To this end, we turn to decorated moduli spaces of rank r vector bundles on C. In addition to r, we think of the *level* k as fixed. We denote by $\mathcal{P}_{r,k}$ the set of Young diagrams with at most r rows and at most k columns. Enumerating the lengths of the rows, we write a diagram λ as

$$\lambda = (\lambda^1, \ldots, \lambda^r),\ k \geq \lambda^1 \geq \cdots \geq \lambda^r \geq 0.$$

Such vectors can also be regarded as highest weights for irreducible representations of the unitary group $U(r)$, bounded by k.

We consider the curve C together with a finite set I of distinct points on it, and partitions $\lambda_p \in \mathcal{P}_{r,k}$ labeled by the points $p \in I$. The lengths of *columns* in a partition $\lambda \in \mathcal{P}_{r,k}$ give a flag type on an r-dimensional vector space. A vector bundle E together with a choice of a flag in each of its fibers over the points in I,

$$0 \subset E_{1,p} \subset E_{2,p} \subset \ldots \subset E_{k,p} = E_p$$

with flag type given for each $p \in I$ by the partition λ_p, is referred to as a *parabolic* vector bundle of type $\underline{\lambda} = (\lambda_p)_{p\in I}$.

The lengths of *rows* in a partition λ_p add the datum of a set of weights to the flag type at p, and define a parabolic slope for E,

$$\mu_{\text{par}}(E) = \frac{d}{r} + \frac{|\underline{\lambda}|}{rk}, \tag{9}$$

with $|\underline{\lambda}|$ being the total number of boxes in all partitions $\lambda_p, p \in I$. As in the case of undecorated bundles, the slope comes with a notion of semistability, and there is a coarse projective moduli space $U_C(r, d, \underline{\lambda})$ of semistable rank r degree d parabolic vector bundles of type $\underline{\lambda}$, introduced in [MS].

The construction is similar to that of the undecorated space $U_C(r, d)$. Its brief description here follows [P]. To start, let Ω be the open locus in the Quot scheme $\text{Quot}_C^{r,d}(\mathcal{O}_C^{\oplus q}(-m))$ where the universal quotient sheaf

$$\mathcal{Q} \to \text{Quot}_C^{r,d}(\mathcal{O}_C^{\oplus q}(-m)) \times C$$

is locally free, and in addition each quotient

$$\mathcal{O}_C^{\oplus q}(-m) \to E$$

in Ω gives an isomorphism

$$H^0(\mathcal{O}_C^{\oplus q}) \simeq H^0(E(m)).$$

For each point $p \in I$, consider next the restriction

$$\mathcal{Q}_p = \mathcal{Q}|_{\Omega\times\{p\}}$$

of the universal quotient bundle, and its associated flag bundle Fl_{λ_p}, where the flag type is specified by the partition λ_p. Let R be the product over Ω of the flag bundles for each $p \in I$,

$$R = Fl_{\lambda_{p_1}} \times_\Omega \cdots \times_\Omega Fl_{\lambda_{p_n}}.$$

The moduli space of semistable parabolic vector bundles of type $\underline{\lambda}$ is the GIT quotient

$$U_C(r, d, \underline{\lambda}) =_{\text{def}} R^{ss}//SL(q),$$

where R^{ss} is the open semistable locus in R defined in terms of the slope (9).

We describe natural theta bundles over $U_C(r, d, \underline{\lambda})$. One can consider on Ω the level k determinant line bundle

$$(\det R\pi_\star(\mathcal{Q}))^{-k},$$

where as usual

$$\pi : \Omega \times C \to \Omega$$

is the projection. Furthermore each flag bundle Fl_{λ_p} carries a natural line bundle

$$\mathcal{N}_p \to Fl_{\lambda_p}$$

restricting fiberwise to the Borel-Weil ample line bundle on the fibers. Concretely, these Borel-Weil line bundles are determinants of universal quotients on the flag bundle. Under the condition

$$kd + |\lambda| \equiv 0 \mod r \tag{10}$$

the tensor product

$$(\det R\pi_\star(\mathcal{Q}))^{-k} \bigotimes_{p \in I} \mathcal{N}_p \otimes (\det Q_x)^e$$

descends to a line bundle

$$\mathcal{L}_{\underline{\lambda}} \ \to \ U_C(r, d, \underline{\lambda})$$

on the GIT quotient. Here x is a point on the curve (which will be omitted from the notation), and

$$e = \frac{kd + |\lambda|}{r} + k(1 - g).$$

When $\underline{\lambda}$ consists of empty partitions, and $d = 0$, we recover the space $U_C(r, 0)$ and the line bundle $\Theta_{r,M}$ where $M = \mathcal{O}((g-1)x)$.

We set

$$V^{r,k}_{g,d}(\underline{\lambda}) = h^0(U_C(r, d, \underline{\lambda}), \mathcal{L}_{\underline{\lambda}} \otimes \det{}^\star \Theta_1) = \chi(U_C(r, d, \underline{\lambda}), \mathcal{L}_{\underline{\lambda}} \otimes \det{}^\star \Theta_1). \tag{11}$$

The case of degree $d = 0$ is particularly important; for simplicity, we write

$$V^{r,k}_g(\underline{\lambda}) = V^{r,k}_{g,0}(\underline{\lambda}).$$

The parabolic Verlinde numbers $V_g^{r,k}(\underline{\lambda})$ are given by explicit elementary formulas similar to (8). Refraining from writing these down, we describe next the relationship between $V_g^{r,k}(\underline{\lambda})$ and intersections on the Quot scheme.

5. The GL Verlinde TQFT at fixed rank and level

5.1. **Euler characteristics and intersections on the Quot scheme.** The theory of Euler characteristics of determinant line bundles over the moduli space $U_C(r,0)$ is naturally related to the intersection theory of the space

$$\mathrm{Mor}_d(C, \mathbb{G}(r, k+r))$$

of degree d maps to $\mathbb{G}(r, k+r)$, where

$$d \equiv 0 \mod r.$$

We discuss this connection, stated and proved in [W], [A], in the next section. One of its most concrete aspects is the following remarkable formula for the undecorated Verlinde numbers. Recall the top Chern class a_r, defined in (3), on the Quot scheme $Q_C(\mathbb{G}(r,k+r), d)$ compactifying $\mathrm{Mor}_d(C, \mathbb{G}(r,k+r))$. We define the integer

$$t = \frac{d}{r}(k+r) - k(g-1),$$

so that the expected dimension of $Q_C(\mathbb{G}(r,n),d)$ equals rt. The Verlinde number $V_g^{r,k}$ can be expressed as a top intersection

$$V_g^{r,k} = \int_{[Q_C(\mathbb{G}(r,k+r),d)]^{vir}} a_r^t. \tag{12}$$

Note that although d is arbitrary divisible by r, Proposition 1 ensures that (12) gives the same answer for different values of d.

It can be easily checked in fact that (12) holds: the Vafa-Intriligator sum giving the right-hand side integral can be immediately written as the elementary formula (8). More satisfyingly, geometric arguments [MO2] relate the intersection theory of the space $U_C(r,d)$ with that of the Quot scheme $Q_C(\mathbb{G}(r,n),d)$ in the large n limit. The particular expression of the Todd class appearing in holomorphic Euler characteristic calculations then recasts the Verlinde number $V_g^{r,k}$ as the intersection (12) on the finite Quot scheme $Q_C(\mathbb{G}(r,k+r),d)$.

An analogue of (12) holds for the decorated degree 0 Verlinde numbers $V_g^{r,k}(\underline{\lambda})$, which are well defined provided that

$$|\underline{\lambda}| \equiv 0 \mod r. \tag{13}$$

To explain the result, we need more notation. To an individual partition $\lambda \in \mathcal{P}_{r,k}$ we associate the Schur polynomial in the Chern roots $x_1, \ldots, x_r$ of the rank r

universal sheaf $\mathcal{S}_p^\vee$:

$$\sigma_\lambda(x_1, \ldots, x_r) = \frac{\det(x_i^{\lambda^j + r - j})}{V(x_1, \ldots, x_r)},$$

where $V(x_1, \ldots, x_r)$ is the Vandermonde determinant. We denote the ensuing class

$$a_\lambda = \sigma_\lambda(\mathcal{S}_p^\vee).$$

For a multipartition $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$, we set

$$a_{\underline{\lambda}} = a_{\lambda_1} \cdots a_{\lambda_n}.$$

Next, to a partition

$$\lambda : k \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 0 \text{ in } \mathcal{P}_{r,k},$$

we associate the conjugate partition $\lambda^\star \in \mathcal{P}_{r,k}$,

$$\lambda^\star : k \geq k - \lambda_r \geq \cdots \geq k - \lambda_1 \geq 0.$$

The definition extends naturally to multipartitions $\underline{\lambda}$.

When (13) holds, we have

$$V_g^{r,k}(\underline{\lambda}) = \int_{[\mathsf{Q}_C(\mathbb{G}(r,k+r),d)]^{vir}} a_{\underline{\lambda}^\star} \cdot a_r^t. \tag{14}$$

Here the degree d is as before any number divisible by r, and t is then taken to satisfy the dimension equation

$$|\underline{\lambda}^\star| + rt = (k+r)d - rk(g-1).$$

The identity (14) can be checked as earlier using the Vafa-intriligator formula to calculate the right-hand side integral, and the Verlinde formula for parabolic bundles in [Bea]. Formulas related to (14) were written down in [O] in the process of establishing a level-rank duality on moduli of parabolic bundles.

5.2. **The Grassmann TQFT.** The Verlinde numbers are the closed invariants

$$F(g) = V_g^{r,k}$$

in a TQFT which we now describe. We refer to this theory equally as the *GL* Verlinde, or the Grassmann TQFT. The theory was introduced in [W], which we follow closely, while expressing the main facts in standard mathematical form. The fundamental vector space of the TQFT, together with a preferred basis, is

$$H = \bigoplus_{\lambda \in \mathcal{P}_{r,k}} \mathbb{C}\lambda.$$

Considering the Grassmannian $G(r, k+r)$ and its tautological sequence

$$0 \to S \to \mathcal{O} \otimes \mathbb{C}^{r+k} \to Q \to 0,$$

we think of $x_1, \ldots, x_r$ as being the Chern roots of the dual tautological bundle $S^\vee$. In this case, the Schur polynomials $\sigma_\lambda(x_1, \ldots, x_r)$ give a basis for the cohomology of the Grassmannian, and we may view

$$H = \bigoplus_{\lambda \in \mathcal{P}_{r,k}} \mathbb{C}\, \sigma_\lambda = H^\star(G(r, k+r), \mathbb{C}).$$

The numbers $F(g)$ were written in the previous section as intersections on a suitable Quot scheme. The general matrix elements of $F(W_s^u(g))$ are integrals on the Quot scheme as well. We consider the Quot schemes for all degrees at once, setting

$$Q_{C,r,k} = \coprod_d Q_C(\mathbb{G}(r, k+r), d).$$

As explained in the previous subsection, they come equipped with natural cohomology classes $a_{\underline{\lambda}}$, indexed by multipartitions. To start, for $\underline{\lambda}$ a multipartition with s components, we define the matrix elements $F(g)_{\underline{\lambda}}$ of the homomorphism

$$F(W_s^0(g)) : H^{\otimes s} \to \mathbb{C}$$

by

$$F(g)_{\underline{\lambda}} = \int_{[Q_{C,r,k}]^{vir}} a_{\underline{\lambda}} \cdot a_r^{rg+k}. \tag{15}$$

We define the matrix elements $F(g)_{\underline{\lambda}}^{\underline{\mu}}$ in full generality by

$$F(g)_{\underline{\lambda}}^{\underline{\mu}} = \int_{[Q_{r,k}]^{vir}} a_{\underline{\lambda}} \cdot a_{\underline{\mu}^\star} \cdot a_r^{r(g+u)+k}, \tag{16}$$

where u is the number of components of the multipartition $\underline{\mu}$. Note that only one summand contributes to the infinite sum (16), since integration only occurs over the Quot scheme of degree

$$d = \frac{|\underline{\lambda}| - |\underline{\mu}|}{k+r} + r(g+u). \tag{17}$$

If this expression does not yield an integer *i.e.*,

$$|\underline{\lambda}| \not\equiv |\underline{\mu}| \mod k+r \tag{18}$$

the matrix element $F(g)_{\underline{\lambda}}^{\underline{\mu}}$ is 0. Letting μ in (16) consist of no partitions, we recover (15). When λ and μ both consist of no partitions, we obtain

$$F(g) = \int_{[Q_{r,k}]^{vir}} a_r^{rg+k}$$

which is a particular case of equation (12) for $d = rg$.

In the last section we show that the numbers $F(g)_{\underline{\lambda}}^{\underline{\mu}}$ satisfy the requisite gluing formula (1) of a TQFT.

Remark 1. *Comparison with the quantum cohomology of* $\mathbb{G}(r, k+r)$. There is a slight asymmetry between the roles of $\underline{\lambda}$ and $\underline{\mu}$ in (16), with only the number of components of the multi-index $\underline{\mu}$ appearing explicitly in the defining integral. This reflects a twist in the metric $F(W_2^0(0))$ on the Frobenius algebra H. The metric is given by

$$(\sigma_\lambda, \sigma_\mu) = F(0)_{\lambda,\mu} = \int_{[Q_{\mathbb{P}^1,r,k}]} a_\lambda\, a_\mu \cdot a_r^k,$$

which manifestly differs from the usual Poincaré pairing

$$\int_{\mathbb{G}(r,k+r)} a_\lambda\, a_\mu.$$

Turning now to the algebra structure on H, we have

$$\sigma_\lambda \cdot \sigma_\mu = \sum_\nu F(0)^\nu_{\lambda,\mu} \sigma_\nu,$$

where

$$F(0)^\nu_{\lambda,\mu} = \int_{[Q_{\mathbb{P}^1,r,k}]} a_\lambda\, a_\mu\, a_{\nu^\star} \cdot a_r^{k+r} = \int_{[Q_{\mathbb{P}^1,r,k}]} a_\lambda\, a_\mu\, a_{\nu^\star}.$$

The last integral gives precisely the structure constants of the quantum multiplication on $H^\star(\mathbb{G}(r, k+r), \mathbb{C})$ in the Schur basis. Therefore, we obtain an *algebra* isomorphism with quantum cohomology

$$H \cong QH^\star(\mathbb{G}(r, k+r)).$$

Being based on the Poincaré metric, the quantum cohomology as a TQFT is different however from the Grassmann TQFT given by the numbers $F(g)^{\underline{\mu}}_{\underline{\lambda}}$. This is accounted for by the disparity between the two metrics.

5.3. **Formulation in terms of Verlinde data.** The closed invariants $F(g)$ coincide with the undecorated Verlinde numbers $V_g^{r,k}$. In general, matrix elements $F(g)_{\underline{\lambda}}$ can be expressed as Verlinde numbers. This is easily checked when

$$|\underline{\lambda}| \equiv 0 \mod r(k+r).$$

Since $|\underline{\lambda}^\star|$ is divisible by r, we have well-defined line bundle

$$\mathcal{L}_{\underline{\lambda}^\star} \otimes \det{}^\star \Theta_1$$

over the moduli space $U_C(r, 0, \underline{\lambda}^\star)$. Letting

$$d = rg + \frac{|\underline{\lambda}|}{k+r},$$

we may apply (14) to conclude

$$F(g)_{\underline{\lambda}} = V_g^{r,k}(\underline{\lambda}^\star).$$

For arbitrary $\underline{\lambda}$, the Verlinde number $F(g)_{\underline{\lambda}}$ is non-zero when

$$|\underline{\lambda}| \equiv 0 \mod k+r,$$

cf. (18). In this situation, for the degree d given by (17):

$$d = \frac{|\underline{\lambda}|}{k+r} + rg,$$

we have

$$kd - |\underline{\lambda}| \equiv 0 \mod r \iff kd + |\underline{\lambda}^{\star}| \equiv 0 \mod r.$$

Then, there is a well-defined line bundle

$$\mathcal{L}_{\underline{\lambda}^{\star}} \to U_C(r, d, \underline{\lambda}^{\star}),$$

and we expect that

$$F(g)_{\underline{\lambda}} = \chi\left(U_C(r, d, \underline{\lambda}^{\star}), \mathcal{L}_{\underline{\lambda}^{\star}} \otimes \det{}^{\star}\Theta_1\right). \tag{19}$$

More generally, we expect the equality

$$F(g)_{\underline{\lambda}}^{\underline{\mu}} = \chi(U_C(r, d, \underline{\lambda}^{\star}, \underline{\mu}), \mathcal{L}_{\underline{\lambda}^{\star}, \underline{\mu}} \otimes \det{}^{\star}\Theta_1), \tag{20}$$

for the degree d as in (17). The parabolic Verlinde numbers for arbitrary degree d have been less explored, but it should be possible to check these claims using the formulas of [J].

Remark 2. *Comparison with the $SU(r)$ level k fusion algebra.* A closely related theory is the well-studied SL Verlinde TQFT described in [Bea] [TUY]. The underlying vector space

$$\widetilde{H} = \bigoplus_{\rho} \mathbb{C}\rho$$

is labeled by heighest weight representations ρ of $SU(r)$ at level k. Most concretely, we think of ρ as equivalence classes of partitions $\lambda \in \mathcal{P}_{r,k}$, where

$$\lambda \sim \mu$$

if λ and μ are obtained from one another by adding or subtracting the same number of boxes from the rows.

In this basis, the matrix elements $\widetilde{F}(g)_{\underline{\lambda}}^{\underline{\mu}}$ of the theory are given as Verlinde numbers

$$\widetilde{F}(g)_{\underline{\lambda}}^{\underline{\mu}} = \chi(\mathcal{L}_{\underline{\lambda}, \underline{\mu}^{\star}})$$

where

$$\mathcal{L}_{\underline{\lambda}, \underline{\mu}^{\star}} \to SU_C(r, \underline{\lambda}, \underline{\mu}^{\star})$$

is the level k determinant bundle over the moduli space of parabolic bundles with trivial determinant. The degeneration formulas, known as factorization rules, were famously proved in [TUY] using the connection with conformal blocks.

The underlying algebra of the theory $\widetilde{F}$ is a quotient of the quantum cohomology of $\mathbb{G}(r, k+r)$ in the available standard presentation of [ST]. This fact is explained for instance in [KS].

5.4. **Degeneration rules.** To prove that the matrix elements $F(g)_{\underline{\lambda}}^{\underline{\mu}}$ satisfy (1), we show the two degeneration formulas

$$F(g)_{\underline{\lambda}}^{\underline{\mu}} = \sum_{\rho \in \mathcal{P}_{r,k}} F(g-1)_{\underline{\lambda}, \rho}^{\underline{\mu}, \rho}, \tag{21}$$

and

$$F(g)_{\underline{\lambda}}^{\underline{\mu}} = \sum_{\rho \in \mathcal{P}_{r,k}} F(g_1)_{\underline{\lambda_1}, \rho}^{\underline{\mu}_1} \cdot F(g_2)_{\underline{\lambda_2}}^{\underline{\mu_2}, \rho} \tag{22}$$

for splittings

$$g = g_1 + g_2,\ \underline{\lambda} = \underline{\lambda}_1 + \underline{\lambda}_2,\ \underline{\mu} = \underline{\mu}_1 + \underline{\mu}_2.$$

The argument is standard. Suppose first that a smooth curve C of genus g degenerates to a nodal irreducible curve C_0 with one node s, and let $\widetilde{C}$ be the smooth genus $g-1$ curve normalizing C_0. We write the class of the diagonal

$$\Delta \subset \mathbb{G}(r, k+r) \times \mathbb{G}(r, k+r)$$

as

$$[\Delta] = \sum_{\rho \in \mathcal{P}_{r,k}} \sigma_\rho(x_1, \ldots, x_r)\, \sigma_{\rho^\star}(x_1', \ldots, x_r'),$$

where the primed variables are the Chern roots of the tautological bundle $S^\vee$ on the second Grassmannian. We denote by ω the Euler class of $\mathbb{G}(r, k+r)$ *i.e.,* the pullback of the diagonal class under the standard embedding,

$$\omega(x_1, \ldots, x_r) = \sum_{\rho \in \mathcal{P}_{r,k}} \sigma_\rho(x_1, \ldots, x_r) \sigma_{\rho^\star}(x_1, \ldots, x_r).$$

For any top polynomial $P(a_1, \ldots, a_r)$ and sufficiently large degrees d, it was shown in [Ber] that

$$\begin{aligned} \int_{Q_C(\mathbb{G}(r,k+r),d)} P(a_1, \ldots, a_r) &= \int_{Q_{\widetilde{C}}(\mathbb{G}(r,k+r),d)} P(a_1, \ldots, a_r)\, \omega(\mathcal{S}_p^\vee) \\ &= \sum_{\rho \in \mathcal{P}_{r,k}} \int_{Q_{\widetilde{C}}(\mathbb{G}(r,k+r),d)} P(a_1, \ldots, a_r)\, a_\rho\, a_{\rho^\star}. \end{aligned} \tag{23}$$

Here we regard ω as a polynomial in the Chern roots of the universal bundle $\mathcal{S}_p^\vee$ on the Quot scheme. Equation (23) expresses the fact that the space of maps $\mathrm{Mor}_d(C_0, \mathbb{G}(r, k+r))$ is embedded in the larger space $\mathrm{Mor}_d(\widetilde{C}, \mathbb{G}(r, k+r))$ as

$$\mathrm{Mor}_d(C_0, \mathbb{G}(r, k+r)) = ev_2^{-1}(\Delta),$$

where ev_2 denotes the evaluation map

$$ev_2 : \mathrm{Mor}_d(\widetilde{C}, \mathbb{G}(r, k+r)) \to \mathbb{G}(r, k+r) \times \mathbb{G}(r, k+r)$$

at the two points s_1 and s_2 over the node of C_0. The intersections are moreover enumerative in the large degree regime. Proposition 1 then ensures that (23) holds in arbitrary degree when the integrals are evaluated against the virtual fundamental class.

If we let C degenerate to a reducible nodal curve with one node and two smooth irreducible components C_1 and C_2 of genera g_1 and g_2, such that

$$g = g_1 + g_2,$$

a similar argument shows

$$\begin{aligned}\int_{[Q_C(\mathbb{G}(r,k+r),d)]^{vir}} P \cdot Q(a_1, \ldots a_r) &= \sum_{\rho \in \mathcal{P}_{r,k}} \sum_{d_1+d_2=d} \int_{[Q_{C_1}(\mathbb{G}(r,k+r),d_1)]^{vir}} P(a_1, \ldots, a_r)\, a_\rho \\ (24) \qquad\qquad &\cdot \int_{[Q_{C_2}(\mathbb{G}(r,k+r),d_2)]^{vir}} Q(a_1, \ldots, a_r)\, a_{\rho^\star}.\end{aligned}$$

Equation (24) is also argued geometrically in the large degree regime, where the intersections involved are enumerative. The passage to arbitrary degree and the virtual fundamental class is again via Proposition 1.

The degeneration rule (21) follows from (23) taking

$$P(a_1, \ldots, a_r) = a_{\underline{\lambda}} \cdot a_{\underline{\mu}^\star} \cdot a_r^{r(g+u)+k},$$

with u the cardinality of the multi-index $\underline{\mu}$. Similarly (22) follows from (24) taking

$$P = a_{\underline{\lambda_1}} \cdot a_{\underline{\mu_1}^\star} \cdot a_r^{r(g_1+u_1)+k}, \quad Q = a_{\underline{\lambda_2}} a_{\underline{\mu_2}^\star} \cdot a_r^{r(g_2+u_2)},$$

with u_1, u_2 being the number of components of $\underline{\mu}_1, \underline{\mu}_2$.

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