

# ON THE KOSTANT MULTIPLICITY FORMULA FOR GROUP ACTIONS WITH NON-ISOLATED FIXED POINTS

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## 1. INTRODUCTION

Let  $M$  be a compact, connected,  $2d$ -dimensional manifold equipped with a symplectic form,  $\omega$ , and with a Hermitian line bundle,  $\mathbb{L}$ , and an almost-complex structure,  $J$ , which are compatible with  $\omega$ . (By *compatible* we mean that

$$(1.1) \quad c(\mathbb{L}) = [\omega]$$

and that the bilinear form

$$(1.2) \quad g_p(v, w) = \omega_p(J_p v, w) ; v, w \in T_p M ,$$

is symmetric and positive definite.) From  $J_p$  one gets a Dolbeault structure on the exterior algebra of  $T_p^*$ :

$$\bigwedge^i(T_p^*) \otimes \mathbb{C} = \sum_{i=j+k} \bigwedge_p^{j,k} .$$

For  $\xi \in T_p^*$ , let  $\xi_{0,1}$  be the  $\bigwedge^{0,1}$ -component of  $\xi$ , and let

$$\gamma_p(\xi): \mathbb{L}_p \otimes \bigwedge_p^{0,i} \rightarrow \mathbb{L}_p \otimes \bigwedge_p^{0,i+1}$$

be the map,

$$\gamma_p(\xi)w = \xi_{0,1} \wedge w .$$

The compatibility condition (1.2) implies that  $g_p$  and  $\omega_p$  are the real and imaginary parts of a Hermitian inner product; and from this inner product and the inner product on  $\mathbb{L}_p$  one gets inner products on the domain and range of  $\gamma_p(\xi)$ . Let  $\gamma_p(\xi)^*$  be the transpose of  $\gamma_p(\xi)$ , and let

$$(1.3) \quad \sigma_p(\xi): \mathbb{L}_p \otimes \bigwedge_p^{0,\text{even}} \rightarrow \mathbb{L}_p \otimes \bigwedge_p^{0,\text{odd}}$$

be the sum of  $\gamma_p(\xi)$  and  $\gamma_p(\xi)^*$ . For  $\xi \neq 0$  this map is bijective; so there exists a first order elliptic differential operator,  $D$ , whose symbol is (1.3). We will denote by  $\text{ind}(D)$  the virtual vector space

$$(1.4) \quad \text{kernel}(D) - \text{cokernel}(D) .$$

Now let  $G$  be a compact connected Lie group, and let  $\tau$  be an effective action of  $G$  on  $M$  which preserves  $\omega$  and  $J$ . We will assume that there is an action,  $\tau_1$ , of  $G$  on  $\mathbb{L}$  which is compatible with  $\tau$  and hence, in particular (see [GS]) that  $\tau$  is a Hamiltonian action with moment map,  $\phi: M \rightarrow \mathfrak{g}^*$ . From  $\tau_1$  one gets an induced action of  $G$  on the sections of  $\mathbb{L} \otimes \bigwedge^{0,*}$  and, by averaging, one can make  $D$  commute with this action. Thus one gets a representation of  $G$  on  $\text{ind}(D)$  which, up

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(\*) Partially supported by INVOTAN/NATO grant 15/A/94/PO. (\*\*) Supported by NSF grant DMS 890771.

to isomorphism, is a *Hamiltonian* invariant of  $M$ , *i.e.*, depends on  $(\tau, \phi)$  but doesn't depend on  $J$  or  $D$ . To compute this invariant, one can, without loss of generality, assume that  $G$  is abelian (see appendix A) in which case this representation is completely determined by its weight multiplicities. If  $M^G$  is finite, these are given by the Kostant multiplicity formula:

$$(1.5) \quad \#(\alpha, \text{ind}(D)) = \sum (-1)^{\sigma_i} N_i(\alpha)$$

the left hand side being the multiplicity of the weight,  $\alpha$ , and  $N_i$  being the ‘‘Kostant partition function’’ associated with the isotropy representation of  $G$  at the  $i$ -th fixed point.<sup>1;2</sup>

In this article we will show that a formula of this type is true when  $M^G$  isn't finite. Let's denote the connected components of  $M^G$  by  $F_i$ ,  $i = 1, \dots, N$ , and let  $\mathbb{N}F_i$  be the normal bundle of  $F_i$ .  $\mathbb{N}F_i$  splits into a direct sum of vector subbundles

$$(1.6) \quad \mathbb{E}_{i,1} \oplus \dots \oplus \mathbb{E}_{i,m} ,$$

$m$  depending on  $i$ , such that the isotropy representation of  $\mathfrak{g}$  on  $\mathbb{E}_{i,j}$  is multiplication by a fixed weight,  $\alpha_{i,j}$  (where  $\alpha_{i,j} \neq \alpha_{i,k}$  for  $j \neq k$ ). We will polarize these weights as in [GLS] by choosing an element,  $v$ , of  $\mathfrak{g}$  such that  $\alpha_{i,j}(v) \neq 0$  for all  $i, j$ , and setting

$$(1.7) \quad \alpha_{i,j}^\# = \epsilon_{i,j} \alpha_{i,j}$$

where

$$(1.8) \quad \epsilon_{i,j} = \text{sign } \alpha_{i,j}(v) .$$

(These polarized weights have the property that they all lie in the half-space  $0 < (\xi, v)$ .) Let  $n_{i,j}$  be the rank of the vector bundle,  $\mathbb{E}_{i,j}$ , and let

$$(1.9) \quad \delta_i = \sum_{\epsilon_{i,j}=-1} n_{i,j} \alpha_{i,j}^\# \quad \text{and} \quad \sigma_i = \sum_{\epsilon_{i,j}=-1} n_{i,j} .$$

For every  $m$ -tuple of non-negative integers,  $k = (k_1, \dots, k_m)$ , let  $\mathbb{E}_i(k)$  be the tensor product

$$(1.10) \quad \left( \bigotimes_{j=1}^m \mathcal{S}^{k_j}(\mathbb{E}_{i,j}^\#) \right) \otimes \left( \bigotimes_{\epsilon_{i,j}=-1} \wedge^{n_{i,j}}(\mathbb{E}_{i,j}^\#) \right)$$

where  $\mathbb{E}_{i,j}^\# = \mathbb{E}_{i,j}$  or  $\mathbb{E}_{i,j}^*$  depending on whether  $\epsilon_{i,j}$  is 1 or  $-1$ . Finally let  $\Delta_i(\alpha)$  be the convex polytope in  $\mathbb{R}^m$  consisting of all  $m$ -tuples,  $(s_1, \dots, s_m)$ ,  $s_i \geq 0$ , for which

$$(1.11) \quad \sum_j s_j \alpha_{i,j}^\# + \phi_i = \alpha$$

where  $\phi_i$  is the value of  $\phi$  on  $F_i$ . (The fact that the  $\alpha_{i,j}^\#$ 's are polarized implies that  $\Delta_i(\alpha)$  is compact.) Our generalization of the Kostant formula is the following:

<sup>1</sup>For the definition of  $N_i$  and  $\sigma_i$ , see below.

<sup>2</sup>This formula was discovered by Kostant [Ko] in the middle fifties in the setting of coadjoint orbits and, in the late eighties, extended by Guillemin, Lerman and Sternberg [GLS] to the setting above.

**Theorem 1.** *The multiplicity with which  $\alpha$  occurs as a weight of the representation of  $G$  on  $\text{ind}(D)$  is equal to the sum (1.5) where*

$$(1.12) \quad N_i(\alpha) = \sum_{k \in \Delta_i(\alpha - \delta_i)} \int_{F_i} \text{Ch}(\mathbb{E}_i(k) \otimes \mathbb{L}) \text{Todd}(F_i)$$

$\text{Todd}(F_i)$  being the Todd class of  $F_i$  (with the almost-complex structure induced on  $F_i$  by  $J$ ) and  $\text{Ch}(\mathbb{E}_i(k) \otimes \mathbb{L})$  being the Chern character of  $\mathbb{E}_i(k) \otimes \mathbb{L}$ .

**Remark:** If  $M^G$  is finite (1.12) reduces to

$$(1.13) \quad N_i(\alpha) = \sum_{k \in \Delta_i(\alpha - \delta_i)} \binom{k_1 + n_{i,1} - 1}{n_{i,1} - 1} \cdots \binom{k_m + n_{i,m} - 1}{n_{i,m} - 1}.$$

The formula (1.12) has an interesting “semi-classical” limit. Replacing the line bundle,  $\mathbb{L}$ , by its  $n$ -th tensor power, one gets, in analogy with (1.3), an elliptic symbol

$$\sigma_p^{(n)}(\xi): \mathbb{L}_p^n \otimes \bigwedge_p^{0,\text{even}} \rightarrow \mathbb{L}_p^n \otimes \bigwedge_p^{0,\text{odd}}.$$

Let  $D_n$  be a  $G$ -invariant elliptic operator with this as its symbol and let  $\gamma = \dim G$ .

**Theorem 2.** *As  $n$  tends to infinity, the quantity  $n^{-(d-\gamma)} \#(n\alpha, \text{ind}(D_n))$  tends to*

$$(1.14) \quad \sum (-1)^{\sigma_i} \int_{\Delta_i(\alpha)} \text{Res}_i(s) ds$$

where  $\text{Res}_i(s)$  is the residue at  $z = 0$  of

$$(1.15) \quad \exp\left(\sum s_j z_j\right) \int_{F_i} \frac{\exp[\omega]}{c_{i,1}(z_1) \cdots c_{i,m}(z_m)}$$

and  $c_{i,j}(z)$  is the Chern polynomial of  $\mathbb{E}_{i,j}^\#$ .

For the case of isolated fixed points (1.14) reduces to:

$$(1.16) \quad \sum (-1)^{\sigma_i} \int_{\Delta_i(\alpha)} \frac{s_1^{n_{i,1}-1} \cdots s_m^{n_{i,m}-1}}{(n_{i,1}-1)! \cdots (n_{i,m}-1)!}.$$

In [GLS] it was proved that the function of  $\alpha$  defined by (1.16) is the Radon-Nikodym derivative

$$(1.17) \quad \frac{d\mu_{DH}}{d\mu_{Leb}}$$

where  $\mu_{DH}$  is the Duistermaat-Heckman measure and  $\mu_{Leb}$  is the standard Lebesgue measure on  $\mathfrak{g}^*$  (suitably normalized). It turns out that the same is true for (1.14):

**Theorem 3.** *The piece-wise polynomial function of  $\alpha$  defined by (1.14) is the Radon-Nikodym derivative, (1.17).*

The results above are true, with small modifications, for orbifolds. In particular, in Theorem 1, the integral

$$\int_{F_i} \text{Ch}(\mathbb{E}_i(k) \otimes \mathbb{L}) \text{Todd}(F_i)$$

is the “Riemann-Roch” number of the vector bundle  $\mathbb{E}_i(k) \otimes \mathbb{L}$ , and the orbifold version of Theorem 1 is true if one replaces this by the *Kawasaki* Riemann-Roch number (see [Ka]).

We will conclude this summary of our results by saying a few words about the proofs: It was shown by Cartier in [Ca] that the Kostant multiplicity formula can be derived from the Weyl character formula by expanding the Weyl denominator into a trigonometric series and computing the coefficient of  $e^{i\alpha}$ .<sup>3</sup> In this article we will show that (1.12) can be derived, by essentially the same argument, from the equivariant index theorem of Atiyah-Segal-Singer for  $\text{spin}^c$ -Dirac (see [AS]). Duistermaat [Du] has recently proved that the orbifold analogue of this theorem is true<sup>4</sup>; and, as a consequence, the proof which we give of Theorem 1 in §2 can easily be adapted to the orbifold setting.<sup>5</sup>

## 2. THE PROOF OF THEOREM 1

The equivariant index theorem says that for  $x \in \sqrt{-1}\mathfrak{g}$ ,  $x$  close to zero, the trace of  $\exp \sqrt{-1}x$  on the virtual vector space (1.4) is equal to the sum over the fixed point components,  $F_i$ , of local traces,  $\chi_{F_i}(x)$ , where

$$(2.1) \quad \chi_{F_i}(x) = e^{\phi_i(x)} \int_{F_i} \frac{e^{[\omega]} \text{Todd}(F_i)}{\prod_j \det(I - \exp(\alpha_{i,j}(x)I + \Omega(\mathbb{E}_{i,j})))}$$

$\Omega(\mathbb{E}_{i,j})$  being the curvature form associated with a connection on  $\mathbb{E}_{i,j}$ . To simplify notation in the paragraph below we omit the subscript  $i$ 's in (2.1) and set  $F_i = F$ ,  $\mathbb{E}_{i,j} = \mathbb{E}_j$ ,  $\alpha_{i,j} = \alpha_j$ ,  $\phi_i = \phi_F$ ,  $\epsilon_{i,j} = \epsilon_j$ , etc. If  $\epsilon_j = -1$ , the  $j$ -th term in the denominator can be rewritten:

$$(-1)^{n_j} e^{n_j \alpha_j(x)} \det \exp \Omega(\mathbb{E}_j) \det \left( I - e^{-\alpha_j(x)} \exp(-\Omega(\mathbb{E}_j)) \right)$$

and if we substitute this into (2.1) and let  $D$  be the line bundle

$$\bigotimes_{\epsilon_j = -1} \wedge^{n_j} (\mathbb{E}_j^\#)$$

we can rewrite (2.1) in “polarized” form

$$(-1)^\sigma e^{(\delta + \phi_F)(x)} \int_F \frac{\exp[\omega] \exp \Omega(D) \text{Todd}(F)}{\prod_j \det(I - e^{\alpha_j^\#(x)} \exp \Omega(\mathbb{E}_j^\#))}.$$

By Theorem 1 of appendix B this can be expanded into an infinite trigonometric series.

$$(2.2) \quad (-1)^\sigma \sum_k c_k e^{k_1 \alpha_1^\# + \dots + k_m \alpha_m^\# + \delta + \phi_F}$$

summed over all non-negative integer  $m$ -tuples,  $k$ , where  $c_k$  is equal to

$$(2.3) \quad \int_F \text{trace} \tau_{k_1}(\exp \Omega(\mathbb{E}_1^\#)) \dots \text{trace} \tau_{k_m}(\exp \Omega(\mathbb{E}_m^\#)) \exp(\omega + \Omega(D)) \text{Todd}(F)$$

<sup>3</sup>This is less trivial than it sounds: There are several ways of expanding the Weyl denominator into a trigonometric series, and for some of these expansions the coefficient of  $e^{i\alpha}$  will be given by a divergent infinite sum.

<sup>4</sup>He has, in fact, proved a somewhat deeper result of which this is a consequence: that the “local” version of Atiyah-Segal-Singer is true for  $\text{spin}^c$ -Dirac.

<sup>5</sup>For a more detailed account of the orbifold versions of theorems 1 to 3, see [CG].

or

$$(2.4) \quad \int_F \text{Ch}(\mathbb{E}(k) \otimes \mathbb{L}) \text{Todd}(F) .$$

(Notice that since the  $\alpha_i^\#$  are polarized, the quantity

$$k_1 \alpha_1^\#(v) + \dots + k_m \alpha_m^\#(v) + \delta(v) + \phi_F(v)$$

tends to  $+\infty$  as  $k_1 + \dots + k_m$  tends to  $+\infty$ . Thus for any constant,  $C$ , there are only a finite number of  $k$ 's for which this quantity is less than  $C$ .) On the other hand, for  $x \in \sqrt{-1} \mathfrak{g}$  the trace of  $\exp \sqrt{-1} x$  on the vector space (1.4) is equal to

$$(2.5) \quad \sum \#(\alpha, \text{ind}(D)) e^{\alpha(x)}$$

and by comparing (2.2) with (2.5) one gets the identity (1.12).

### 3. THE PROOF OF THEOREM 2

By Theorem 1,  $\#(n\alpha, \text{ind}(D_n))$  is equal to the sum

$$(3.1) \quad \sum (-1)^{\sigma_i} N_i^{(n)}(n\alpha)$$

where

$$(3.2) \quad N_i^{(n)}(n\alpha) = \sum_k \int_{F_i} \text{Ch}(\mathbb{E}_i(k) \otimes \mathbb{L}^n) \text{Todd}(F_i)$$

summed over all non-negative integral solutions,  $k$ , of the equation

$$(3.3) \quad k_1 \alpha_{i,1}^\# + \dots + k_m \alpha_{i,m}^\# + \delta_i + n\phi_i = n\alpha .$$

(Notice that if we replace  $\mathbb{L}$  by  $\mathbb{L}^n$  we must replace  $\omega$  by  $n\omega$  and  $\phi$  by  $n\phi$ .) As in §2 we will omit all subscript  $i$ 's from now on and let  $F_i = F$ ,  $\alpha_{i,j} = \alpha_j$ , etc. Let  $2p = \dim F$  and  $q = \dim \Delta(\alpha)$ . By (3.2)

$$n^{-(d-\gamma)} N^{(n)}(n\alpha) = n^{-(d-\gamma-p)} \sum_k \int_F n^{-p} \text{Ch}(\mathbb{E}(k) \otimes \mathbb{L}^n) \text{Todd}(F)$$

which is equal to

$$(3.4) \quad n^{-(d-\gamma-p)} \sum_k \int_F \exp[\omega] \text{trace} \tau_{k_1}(\exp \Omega(\mathbb{E}_1^\#)/n) \dots \text{trace} \tau_{k_m}(\exp \Omega(\mathbb{E}_m^\#)/n)$$

up to an error of order  $O(\frac{1}{n})$ . (Proof: With  $\omega$  replaced by  $n\omega$  in (2.3), the integrand in this expression can be expanded into a sum of terms of the form

$$n^{-p} (n\omega)^r \wedge \Omega_{i_1} \wedge \dots \wedge \Omega_{i_s} \wedge \Omega(D)^l \wedge T_\mu$$

where  $\Omega_{i_a}$  is a coefficient of the curvature form,  $\Omega(\mathbb{E}_{i_a}^\#)$ , and  $T_\mu$  is the component of degree  $2\mu$  of  $\text{Todd}(F)$ . However this term can only contribute to the integral if  $r + s + l + \mu = p$  in which case it can be rewritten as

$$\omega^r \wedge (\Omega_{i_1}/n) \wedge \dots \wedge (\Omega_{i_s}/n) \wedge (\Omega(D)/n)^l \wedge T_\mu/n^\mu .$$

Moreover, the terms in this sum for which  $l$  or  $\mu$  is positive can be discarded since they contribute errors of order  $O(\frac{1}{n})$ .

By Theorem B of Appendix B, (3.4) is equal, up to an error of order  $O(\frac{1}{n})$ , to

$$n^{-q} \operatorname{Res}_{z=0} \sum_k e^{\frac{k_1}{n} z_1 + \dots + \frac{k_m}{n} z_m} \int_F \frac{\exp[\omega]}{\det(z_1 I - \Omega(\mathbb{E}_1^\#)) \dots \det(z_m I - \Omega(\mathbb{E}_m^\#))}$$

summed over all  $k$  satisfying

$$\frac{k_1}{n} \alpha_1^\# + \dots + \frac{k_m}{n} \alpha_m^\# + \phi_F + \frac{\delta}{n} = \alpha$$

and as  $n$  tends to infinity this tends to the integral

$$\int_{\Delta(\alpha)} \operatorname{Res}_{z=0} e^{sz} \left( \int_F \frac{\exp[\omega]}{c_{\mathbb{E}_1^\#}(z_1) \dots c_{\mathbb{E}_m^\#}(z_m)} \right) ds .$$

#### 4. THE PROOF OF THEOREM 3

By definition the Duistermaat-Heckman measure is the “push-forward” by the moment map of the symplectic measure on  $M$ , *i.e.*, for a Borel subset,  $B$ , of  $\mathfrak{g}^*$

$$\mu_{DH}(B) = \int_{\phi^{-1}(B)} \frac{\omega^d}{d!} .$$

The inverse Fourier transform of  $\mu_{DH}$  is the function

$$\check{\mu}_{DH}(x) = \int_M e^{\sqrt{-1}(\phi, x)} \frac{\omega^d}{d!}$$

and by the “exact stationary phase” formula [DH] this is equal to the sum over fixed point components

$$(4.1) \quad \sum_i e^{\sqrt{-1} \phi_i(x)} \int_{F_i} \frac{\exp[\omega]}{\prod_j \det(\sqrt{-1} \alpha_{ij}(x) I + \Omega(\mathbb{E}_{ij}))}$$

provided  $\alpha_{ij}(x) \neq 0$  for all  $i$  and  $j$ . Dropping the subscript  $i$ 's and setting  $y = \sqrt{-1} x$ , the  $i$ -th summand becomes

$$(4.2) \quad e^{\phi_F(y)} \int_F \frac{\exp[\omega]}{\prod_j \det(\alpha_j(y) I + \Omega(\mathbb{E}_j))}$$

or

$$(4.3) \quad (-1)^\sigma e^{\phi_F(y)} \int_F \frac{\exp[\omega]}{\prod_j \det(\alpha_j^\#(y) + \Omega(\mathbb{E}_j^\#))} .$$

By Theorem 1 of Appendix B this is equal to

$$(4.4) \quad \frac{(-1)^\sigma e^{\phi_F(y)}}{\prod_j \alpha_j^\#(y)^{n_j}} \sum_{k=0}^{\infty} \frac{1}{\prod_j \alpha_j^\#(y)^{k_j}} \int_F e^{[\omega]} \prod_j \operatorname{trace} \tau_{k_j} \left( -\Omega(\mathbb{E}_j^\#) \right) .$$

(Note that this sum is finite. The terms on the right are zero if  $2 \sum n_j k_j > \dim F$ .) By the Fourier inversion formula the Radon-Nikodym derivative (1.17) is the Fourier transform of (4.4), and we can compute this by computing the Fourier transforms of the summands in (4.4) and summing over  $k$  and the fixed point components. By formula C9 of Appendix C, the Fourier transform of

$$\frac{e^{\phi_F(y)}}{\prod_j \alpha_j^\#(y)^{k_j + n_j}} , \quad y = \sqrt{-1} x ,$$

is the function

$$(4.5) \quad f_k(\alpha) = \int_{\Delta(\alpha)} \frac{s_1^{k_1+n_1-1}}{(k_1+n_1-1)!} \cdots \frac{s_m^{k_m+n_m-1}}{(k_m+n_m-1)!} ds .$$

Substituting this into (4.4) one gets

$$(4.6) \quad (-1)^\sigma \int_{\Delta(\alpha)} ds \left( \int_F e^{[\omega]} \prod_j \frac{s_j^{k_j+n_j-1}}{(k_j+n_j-1)!} \text{trace} \tau_{k_j}(-\Omega(\mathbb{E}_j^\#)) \right) .$$

However, by formula B3 of Appendix B,

$$(4.7) \quad \text{trace} \tau_{k_j}(-\Omega(\mathbb{E}_j^\#)) = \frac{1}{2\pi i} \int_{\Gamma_j} \frac{z_j^{n_j+k_j-1}}{\det(z_j I + \Omega(\mathbb{E}_j^\#))}$$

$\Gamma_j$  being a small contour about the origin in the  $z_j$  plane. If  $k_j < 0$  the integral on the right is zero, so by substituting (4.7) into (4.6) and summing over all  $k_j \geq 0$  (or, equivalently, over all  $k_j + n_j - 1 \geq 0$ ) one gets for the Fourier transform of (4.4):

$$(4.8) \quad (-1)^\sigma \int_{\Delta(\alpha)} ds \left( \text{Res}_{z=0} e^{sz} \int_F \frac{\exp[\omega]}{\prod_j C_{\mathbb{E}_j^\#}(z_j)} \right) .$$

#### APPENDIX A

By the “shifting trick” (see [GS], §6) it suffices to compute the multiplicity with which the trivial representation occurs in the representation of  $G$  on the space (1.4) and (as was pointed out to us by Michèle Vergne) this can easily be computed from the weight multiplicities of the representation of the Cartan subgroup,  $T$ , of  $G$  on the space (1.4). More explicitly the following result is true: Let  $G$  be a compact semi-simple Lie group and  $\rho: G \rightarrow U(Q)$  a representation of  $G$  on a finite dimensional Hilbert space,  $Q$ . Restricting  $\rho$  to  $T$ ,  $Q$  breaks up into weight spaces

$$Q_\xi \quad , \quad \xi \in \mathbb{Z}_T$$

( $\mathbb{Z}_T$  being the weight lattice of  $T$ ). Then

$$(A1) \quad \dim Q^G = \frac{1}{|W|} \sum C_\xi \dim Q_\xi ,$$

the  $C_\xi$ 's being the Fourier coefficients of the function,  $\prod_{\alpha \in \Delta} (1 - e^{i\alpha})$ . In other words,

$$(A2) \quad \prod_{\alpha \in \Delta} (1 - e^{i\alpha(x)}) = \sum_{\xi} C_\xi e^{i\xi(x)} \quad , \quad x \in \mathfrak{t} .$$

(Here  $\Delta$  is the set of roots of  $G$ .)

**Proof.** (A1) can be extracted from the following result of Weyl (see [He], page 194, corollary 5.16).

**Theorem .** *Let  $\chi \in C^\infty(G)$  be a class function (i.e.,  $\chi(aga^{-1}) = \chi(g)$  for all  $a$  and  $g$ .) Then*

$$(A3) \quad \int_G \chi(g) dg = \frac{1}{|W|} \int_T \theta(x) \chi(x) dx$$

$dg$  and  $dx$  being Haar measures on  $G$  and  $T$ ,  $\theta(x)$  being the function (A2) and  $|W|$  being the cardinality of the Weyl group.

**Comments.**

1.  $\theta(x)$  is real and non-negative, as one can see by writing it as the product of the function

$$(A4) \quad \prod_{\alpha \in \Delta_+} (1 - e^{i\alpha})$$

times its conjugate. In particular,  $\theta = \bar{\theta}$ , i.e.,  $C_\xi = C_{-\xi}$ .

2. Let  $\delta$  be half the sum of the positive roots. It is clear from (A4) that  $C_\xi \neq 0 \Rightarrow \xi/2$  lies in the convex hull of  $\{w\delta, w \in W\}$ .

Let's apply (A3) to the character,  $\chi$ , of representation  $\rho$ . Noting that for  $x \in \mathfrak{t}$ :

$$(A5) \quad \chi(\exp x) = \sum e^{i\xi(x)} \dim Q_\xi$$

one gets, by Schur's lemma

$$(A6) \quad \dim Q^G = \int \chi(g) dg = \langle \chi, 1 \rangle_{L^2}$$

(1 being the character of the trivial representation), and hence, by (A3) and (A4)

$$\begin{aligned} \dim Q^G &= \frac{1}{|W|} \int \chi(\exp x) \theta(x) dx \\ &= \frac{1}{|W|} \int \left( \sum_{\xi} e^{i\xi(x)} \dim Q_\xi \right) \left( \sum_{\xi} C_\xi e^{-i\xi(x)} \right) dx \\ &= \frac{1}{|W|} \sum C_\xi \dim Q_\xi . \end{aligned}$$

## APPENDIX B

Let  $V$  be a  $d$ -dimensional vector space over the complex numbers and let  $\tau_k$  be the standard representation of  $GL(V)$  on the  $k$ -th symmetric product,  $\mathcal{S}^k(V)$ .

**Theorem (B1).** For  $z \in \mathbb{C}$ ,  $z$  large, and  $B \in GL(V)$ ,

$$(B1) \quad \det(z - B)^{-1} = z^{-d} \sum_{k=0}^{\infty} z^{-k} \text{trace} \tau_k(B)$$

**Proof.** Without loss of generality we can assume that  $B$  is diagonalizable with eigenvalues,  $\lambda_1, \dots, \lambda_d$ ; in which case the left hand side of (B1) becomes

$$(B2) \quad z^{-d} \prod_{j=1}^d (1 - \lambda_j z^{-1})^{-1} .$$

Expanding each of the factors  $(1 - \lambda_j z^{-1})^{-1}$  into a geometric series one can rewrite (B2) in the form

$$z^{-d} \left( \sum z^{-k} t_k \right)$$

where

$$t_k = \sum_{|I|=k} \lambda_1^{i_1} \cdots \lambda_d^{i_d},$$

and the right hand side of this expression is  $\text{trace}\tau_k(B)$ .

Q.E.D.

**Corollary .** *Let  $\Gamma$  be a contour about the origin containing the zeroes of  $\det(z - B)$ . Then*

$$(B3) \quad \frac{1}{2\pi i} \int_{\Gamma} z^{d+k-1} \det(z - B)^{-1} dz = \text{trace}\tau_k(B).$$

**Remark.** By analyticity, (B1) and (B3) are valid for any endomorphism,  $B: V \rightarrow V$ ; *i.e.*,  $B$  doesn't necessarily have to be in  $GL(V)$ .

From (B3) we will deduce the following useful estimate:

**Theorem (B2).** *Let  $A$  be an endomorphism of  $V$ . Then*

$$(B4) \quad n^{-(d-1)} \text{trace}\tau_k(\exp A/n) = \frac{1}{2\pi i} \left( \int_{\Gamma} e^{\left(\frac{d+k-1}{n}\right)z} \det(z - A)^{-1} dz \right) \left( 1 + O\left(\frac{1}{n}\right) \right)$$

the  $O\left(\frac{1}{n}\right)$  being uniform in  $k$ .

**Proof.** Without loss of generality we can assume that  $A$  is diagonalizable with eigenvalues  $\mu_1, \dots, \mu_d$  and that  $e^{\mu_1}, \dots, e^{\mu_d}$  are distinct. Then by (B3)  $\text{trace}\tau_k(\exp A/n)$  is equal to the contour integral

$$\frac{1}{2\pi i} \int_{\Gamma} z^{d+k-1} (z - e^{\mu_1/n})^{-1} \cdots (z - e^{\mu_d/n})^{-1} dz$$

which, by the residue formula, is equal to

$$\sum_{i=1}^d e^{(d+k-1)\mu_i/n} \prod_{j \neq i} (e^{\mu_i/n} - e^{\mu_j/n})^{-1}$$

or

$$n^{d-1} \left( \sum_{i=1}^d e^{(d+k-1)\mu_i/n} \prod_{j \neq i} (\mu_i - \mu_j)^{-1} \right) \left( 1 + O\left(\frac{1}{n}\right) \right)$$

and, again by the residue formula, this is equal to:

$$n^{d-1} \left( \frac{1}{2\pi i} \int_{\Gamma} e^{\frac{d+k-1}{n}z} \prod_{i=1}^d (z - \mu_i)^{-1} \right) \left( 1 + O\left(\frac{1}{n}\right) \right).$$

Dividing by  $n^{d-1}$  and replacing  $\prod(z - \mu_i)$  by  $\det(z - A)$  we obtain (B4).

## APPENDIX C

Let  $\alpha_1, \dots, \alpha_d$  be vectors in  $\mathbb{R}^n$  which are "polarized" in the sense that, for some  $v \in \mathbb{R}^n$ , the inner products,  $(\alpha_i, v)$ , are all positive. Given  $\phi \in \mathbb{R}^n$  consider the function

$$(C1) \quad e^{i(\phi, x)} \prod_{j=1}^d (\alpha_j, x)^{-1}.$$

Since this function isn't well-defined on all of  $\mathbb{R}^n$ , its Fourier transform is also not well-defined. However, there is a unique measure,  $\mu$ , on  $\mathbb{R}^n$ , with the following two properties:

1. The inverse Fourier transform of  $\mu$  is equal to (C1) on the set

$$(\alpha_j, x) \neq 0 \quad , \quad j = 1, \dots, d .$$

2.  $\mu$  is supported in the half space

$$(\xi, v) \geq (\phi, v) .$$

**Proof.** One can take for  $\mu$  the measure

$$(C2) \quad H_{\alpha_1} * \dots * H_{\alpha_d} * \delta_\phi$$

where  $\delta_\phi$  is the delta-measure at  $\phi$  and

$$(C3) \quad H_{\alpha_i}(f) = \int_0^\infty f(t\alpha_i) dt$$

for continuous functions of compact support,  $f$ .

Q.E.D.

Another description of this measure is the following: Let

$$\mathbb{R}_+^d = \{(s_1, \dots, s_d), s_i \geq 0\}$$

be the positive orthant in  $\mathbb{R}^d$  and let  $L: \mathbb{R}_+^d \rightarrow \mathbb{R}^n$  be the map

$$L(s_1, \dots, s_d) = \sum s_i \alpha_i + \phi .$$

The assumption that the  $\alpha_i$ 's are polarized implies that this is a proper mapping so the measure

$$(C4) \quad L_* ds_1 \dots ds_d$$

is well-defined.

**Theorem (C1).** *The measures (C2) and (C4) are equal.*

**Proof.** In the special case of  $\mathbb{R}^n = \mathbb{R}^d$  and  $\alpha_i = e_i$  (the  $i$ -th standard basis vector) this is just the Fubini theorem. Thus one can write Lebesgue measure on  $\mathbb{R}_+^d$  as the convolution product

$$H_{e_1} * \dots * H_{e_d} .$$

The theorem follows from the fact that  $L_*(H_{e_i}) = H_{\alpha_i}$ , and the fact that, for any pair of measures,  $\mu_1$  and  $\mu_2$ , with support in  $\mathbb{R}_+^d$ ,

$$L_*(\mu_1 * \mu_2) = L_*(\mu_1) * L_*(\mu_2) .$$

Q.E.D.

**Corollary (C2).** *If the vectors,  $\alpha_1, \dots, \alpha_d$ , span  $\mathbb{R}^n$  the measure (C2) is absolutely continuous with respect to Lebesgue measure.*

**Proof.** It suffices to prove that the set of critical points of the map,  $L$ , is of measure zero, which will be the case if and only if  $\alpha_1, \dots, \alpha_d$  span  $\mathbb{R}^n$ . Q.E.D.

Thus, if these hypotheses are satisfied, one can write the measure (C2) in the form

$$(C5) \quad f(\xi) d\xi_1 \dots d\xi_n$$

the function  $f$  being in  $L^1_{loc}$ . In fact it is easy to see that, up to a scalar multiple,<sup>6</sup>

$$(C6) \quad f(\xi) = \text{volume } \Delta(\xi) ,$$

$\Delta(\xi)$  being the convex polytope:

$$(C7) \quad \{s \in \mathbb{R}_+^d, \sum s_i \alpha_i + \phi = \xi\} .$$

By abuse of notation we will refer to (C6) as the *Fourier transform* of the function (C1). Let us compute, in the same spirit, the Fourier transform,  $g$ , of the function

$$(C8) \quad e^{i(\phi, x)} \prod_{j=1}^d (\alpha_j, x)^{-N_j} .$$

Letting  $N = N_1 + \dots + N_d$ , it follows from what we've just proved that  $g(\xi)$  is the volume of the polytope consisting of all  $N$ -tuples

$$t = (t_{1,1}, \dots, t_{1,N_1}, \dots, t_{d,1}, \dots, t_{d,N_d})$$

in  $\mathbb{R}_+^N$  satisfying

$$\sum_{i=1}^d \left( \sum_{j=1}^{N_i} t_{i,j} \right) \alpha_i + \phi = \xi .$$

Let's denote this polytope by  $\tilde{\Delta}(\xi)$ . From the mapping

$$\mathbb{R}_+^N \rightarrow \mathbb{R}_+^d \quad , \quad s_i = \sum_{j=1}^{N_i} t_{i,j} ,$$

one gets a fibration of  $\tilde{\Delta}(\xi)$  over  $\Delta(\xi)$ , the volume of the fiber over  $s$  being

$$\frac{s_1^{N_1-1}}{(N_1-1)!} \dots \frac{s_d^{N_d-1}}{(N_d-1)!} .$$

Hence

$$(C9) \quad g(\xi) = \text{volume } \tilde{\Delta}(\xi) = \int_{\Delta(\xi)} \frac{s_1^{N_1-1}}{(N_1-1)!} \dots \frac{s_d^{N_d-1}}{(N_d-1)!} ds .$$

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<sup>6</sup>By an appropriate normalization of Lebesgue measure in the space,  $\sum s_i \alpha_i = 0$ , one can make this scalar equal to one.

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